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# Mathematical Reviews

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# MATHEMATICAL REVIEWS

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# Mathematical Reviews

Vol. 22, No. 7B

July, 1961

Reviews 5995-6660

## PROBABILITY

See also A5818, A5955, A5956, 6346, 6635, 6638.

5995:

Fisz, Marek. ★*Rachunek prawdopodobieństwa i statystyka matematyczna* [Theory of probability and mathematical statistics]. 2nd ed. revised and enlarged. Biblioteka Matematyczna. Tom 18. Państwowe Wydawnictwo Naukowe, Warsaw, 1958. 530 pp. (1 insert)

The Polish original of the German translation reviewed below.

5996:

Fisz, Marek. ★*Wahrscheinlichkeitsrechnung und Mathematische Statistik*. Hochschulbücher für Mathematik, Bd. 40. VEB Deutscher Verlag der Wissenschaften, Berlin, 1958. x+528 pp. DM 36.00.

The present book is the German translation of the 2nd edition of the author's Polish book [5995 above]. It differs from the 1st edition [1954] to such an extent that it is practically a different book. Most of the shortcomings noted in the review of the first edition [MR 16, 492] have been removed, the contents have been considerably enlarged, the presentation uses modern concepts and terminology and includes a number of topics which have been recently gaining in importance.—As before, there are two parts, the first dealing with probability theory, the second with mathematical statistics. In Part I probability is introduced axiomatically, following Kolmogorov's approach, and random variables are defined as functions on an outcome space; the random variables actually treated are all either discrete or have probability densities. The limit theorems for characteristic functions and random variables are derived in considerable generality; to do this the author has to define and discuss briefly the Stieltjes integral. The presentation of the laws of large numbers includes Kolmogorov's form of the strong law of large numbers. The law of the iterated logarithm is not mentioned. Part I concludes with two chapters, new in this edition, one on Markov chains and another entitled "Stochastic processes", which constitute a clear and useful introduction to the subject.—Part II contains first the usual material on sample moments, chi-square distribution, the Student  $t$  and other statistics based on samples from one- and more-dimensional normal random variables. Then follows a chapter on order statistics containing among others the theory of a number of important distribution-free statistics: Wilks' tolerance limits, the statistics of Kolmogorov, Smirnov and Rényi, and the multi-sample statistics investigated by the author in some of his publications. A separate chapter is devoted to each of the following topics: theory of runs; general theory of

estimation; sampling techniques; fundamentals of analysis of variance; testing hypotheses; elements of sequential analysis. Concise tables of the following distributions are appended: Poisson, normal, chi-square, Student's  $t$ , Fisher's  $Z$ , Kolmogorov-Smirnov limit distribution.—The mathematical level of the book is consistently very good. Except for a few difficult theorems stated without proof (but with references where proofs can be found), all derivations are presented with great care, the exposition is lucid, the arrangement of topics well motivated and their selection such that a student who has read the book will find himself well introduced to most of the important areas of probability and statistics.

Z. W. Birnbaum (Seattle, Wash.)

5997:

Вентцель, Е. С. [Ventcel', E. S.]. ★*Теория вероятностей*. [The theory of probability.] Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 464 pp. (1 insert) 9.55 rubles.

This is a text-book for persons with the mathematical background usually acquired in technical schools and with an interest in the technical applications of probability, particularly in the theory of artillery fire. The theory of stochastic processes receives especial attention.

5998:

Kemeny, John G.; Snell, J. Laurie. ★*Finite Markov chains*. The University Series in Undergraduate Mathematics. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1960. viii+210 pp. \$5.00.

The subject of finite Markov chains can take its place in the undergraduate curriculum equally well as such subjects as "determinants and matrices", "projective geometry", "differential equations" and "statistics". It can be taught to a class with little background, and it has the right combination of notions and techniques for a mathematical education, unlike some of the courses mentioned above. It has contact with several important fields, and if taught properly, always seems to stir interest in the beginner, even without such infantile attractions as the "Land of Oz". Last and not least, it has applications.

The present book has apparently been written with this aim in mind, namely, to provide an alternative or addition to the usual curriculum. There are several books on the subject, notably the treatise by Fréchet [*Recherches théoriques modernes sur le calcul des probabilités*, Gauthier-Villars, Paris, 1938] and Romanovskii's *Diskretnye cepi Markova* [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1949; MR 11, 445]. The book under review is, besides being in English, much more suited for the purpose. It puts beyond its scope such theorems as the central limit theorem, but it treats many others not included in the

cited books, and where they overlap the present treatment is neater. I cite these two other books because there is no comparison with the literature in the book. This may be all right for an undergraduate text on calculus, but not really on a new subject material such as Markov chains.

The scholarly feature of the book consists mainly in a systematic approach by means of symbolic matrix calculus. A basic tool is the matrix formula  $(I-P)^{-1} = \sum_{n=0}^{\infty} P^n$  if  $P^n \rightarrow 0$ . Another device is the consistent use of diagonal matrices (which is, however, not always easy on the eyes). Such symbolic manipulations lead to clean derivations whose results are often recorded as matrix identities. The powerful eigen-value methods of Frobenius and Perron, which form the basis of the usual algebraic treatment, are not discussed. The fact is that the eigen-expansion of  $P^n$  makes transparent nearly all the limiting properties of finite chains. To ignore this is like ignoring the power series expansions of the trigonometrical functions in a treatment of trigonometry.

Among new things may be mentioned: the result, in § 4.4, that the mean first-passage times for distinct pairs uniquely determine the transition matrix; the detailed discussion of "lumpability"; and the applications to learning model and other socio-economic models. The credulous undergraduate should be warned by the instructor that the abundant numerical examples and exercises serve only as drill practice with arithmetic, and need not have any practical significance except in the games of chance, because both the Markovian model and the given data are often too approximate to be taken seriously. (In the reviewer's opinion the back dust-cover contains the kind of advertisement which is not becoming to the profession.)

K. L. Chung (Syracuse, N.Y.)

5999:

Barton, D. E.; David, F. N. Combinatorial extreme value distributions. *Mathematika* 6 (1959), 63-76.

If  $R$  is a discrete random variable, with range from 0 to  $N$  inclusive, and factorial moments  $\{\mu_{(j)}\}$ , then it is known that

$$P(R=r) = \frac{1}{r!} \sum_{j=0}^{N-r} \frac{(-1)^j}{j!} \mu_{(r+j)}.$$

This can be written as

$$P(R=r) =$$

$$\frac{1}{r!} \left\{ \mu_{(r)} - \frac{\mu_{(r+1)}}{1!} + \cdots + \frac{(-1)^{t-r-1}}{(t-r-1)!} \mu_{(t-1)} \right\} + R_t^{(r)},$$

where  $0 \leq (-1)^{t-r} R_t^{(r)} \leq \mu_{(t)}/(t-r)!r!$ .

Suppose  $X_1, \dots, X_n$  are jointly distributed random variables; define  $Z$  as  $\max(X_1, \dots, X_n)$ , and define  $P^*(m)$  as  $P(Z \leq m)$ . Defining  $R$  as the number of variables  $X_1, \dots, X_n$  which are above  $m$  in value,  $R$  is a discrete random variable with  $P^*(m) = P(R=0)$ . The authors apply the formula for  $P(R=r)$  given above to study the function  $P^*(m)$  in the following cases. (a)  $X_1, \dots, X_n$  are independent and identically distributed. (b)  $X_1, \dots, X_n$  are the lengths of the subintervals into which a line segment is broken by points chosen at random. (c)  $X_1, \dots, X_n$  are the lengths of the runs of white balls when balls of various colors are arranged on a line in random order. (d)  $X_1, \dots, X_n$  are the lengths of the runs of the various colors when balls of various colors are arranged on a line in random order. (e)  $X_1, \dots, X_n$  are

multinomial cell frequencies. (f)  $X_1, \dots, X_n$  are the lengths of the runs in ascending or descending order in a sequence of independent, identically distributed random variables.

L. Weiss (Ithaca, N.Y.)

6000:

Anselone, Philip M. Persistence of an effect of a success in a Bernoulli sequence. *J. Soc. Indust. Appl. Math.* 8 (1960), 272-279.

The problem considered is presented as a discrete analogue of the Geiger-Muller counter problem. From a Bernoulli sequence with success probability  $p$ , a sequence  $\{T_n\}$  is constructed by the rule:  $T_n = 1$  if there is at least one success at trials numbered  $n, n-1, \dots, n-h+1$ ;  $T_n = 0$  otherwise. The parameter  $h$  is a measure of persistence of effect and is analogous to counter dead time. The sequence  $\{T_n\}$  is first analyzed with respect to runs of zeroes and ones and then with respect to the number of cycles (two consecutive runs) in a fixed number of trials. The analysis is by means of the theory of recurrent events [W. Feller, *Trans. Amer. Math. Soc.* 67 (1949), 98-119; MR 11, 255]. For the analysis by number of cycles in  $k$  trials,  $N_k$ , the final result is a generating function for the  $r$ th ordinary moment:  $\sum E(N_k^r)z^k$ . Finally it is shown that the limiting value for increasing  $n$  of  $P_r(T_n=0)$  is  $(1-p)^h$ . Applications to radar jamming and to a quality control procedure are described. (Reviewer's note: The numbers  $C_{ri}$  introduced as coefficients of the system of equations  $j^r = \sum_{i=0}^j C_{ri} \binom{j+i}{i}$  are closely related to Stirling numbers; indeed  $C_{ri} = (-1)^{r+i} i! S(r+1, i+1)$ , with  $S(r, i)$  the Stirling number of the second kind.)

J. Riordan (New York)

6001:

Marcus, M. B. Recurrent events in a Bernoulli sequence. *Trans. IRE IT-5* (1959), 179-183.

Let  $\{f_t, t=1, 2, \dots\}$  be a Bernoulli process, i.e., a sequence of independent and identically distributed random functions on a probability space  $(\Omega, \mathcal{B}, P)$  taking the values 0, 1 and such that  $P\{f_t^{-1}(1)\} = p$ ,  $P\{f_t^{-1}(0)\} = 1-p = q$ ,  $0 \leq p \leq 1$ . A point  $\omega \in \Omega$  determines a "time-sequence" (or "sample-sequence")  $(x_t, t=1, 2, \dots)$ ,  $x_t = f_t(\omega)$  of the process. An "event"  $E$  is defined as a sequence  $(\xi_1, \dots, \xi_j)$  of 0's and 1's.

Definitions: (a) Relative to a given event  $E$ , we say that " $\omega$  is in state  $r$  at time (or trial)  $\tau$ ", where  $1 \leq r \leq j$ , if and only if

$$(f_{\tau-r+1}(\omega), \dots, f_{\tau}(\omega)) = (\xi_1, \dots, \xi_r).$$

(b) In case  $r=j$ , we simply say that "for  $\omega$ ,  $E$  occurs at  $\tau$ ".  
(c) If  $\xi_1 = \dots = \xi_j$ , and  $f_{\tau}(\omega) \neq \xi_1$ , we say that " $\omega$  is in state 0 at  $\tau$ ".

The paper describes a simple and direct method of computing the probability for  $k$  occurrences of  $E$  in  $n$  trials, taking  $\xi_1 = \dots = \xi_j = 1$ , and interpreting the term "occurrence" in two ways: (i) to allow possible overlapping of instances of  $E$ , and (ii) to disallow such overlapping. More fully, the following probabilities are sought: (i)  $P\{\omega: \exists \tau_1, \dots, \tau_k, 1 \leq \tau_1 < \dots < \tau_k \leq n, \text{ and, for } \omega, E \text{ occurs at } \tau \Leftrightarrow \tau = \tau_1, \dots, \tau_k\}$ ; (ii)  $P\{\omega: \exists \tau_1, \dots, \tau_k, 1 \leq \tau_1 < \dots < \tau_k \leq n, \tau_k - \tau_{k-1} \geq j, \text{ and, for } \omega, E \text{ occurs at } \tau \Leftrightarrow \tau = \tau_1, \dots, \tau_k\}$ . It is stipulated that for  $\omega$ , the occurrence of  $E$  at  $\tau$  places  $\omega$  in a definite state  $i$  ( $0 \leq i \leq$

$j-1$  at  $\tau$ , as far as the future is concerned. If we take  $i=0$ , we exclude overlapping instances of  $E$  (case (ii) above). If we take  $i \geq 0$  we allow overlapping of instances of  $E$ , the degree of overlapping tolerated depending on  $i$ . For  $i=j-1$ , we get case (i) above.

It is shown that if  $P_r(n, k) = P\{\omega: \text{for } \omega, E \text{ occurs } k \text{ times during the interval } [t, t+n]\omega \text{ is in state } r \text{ at } t\}$ , then

$$P_r(n, k) = pP_{r+1}(n-1, k) + qP_0(n-1, k), \quad 0 \leq r \leq j-2, \\ = pP_r(n-1, k-1) + qP_0(n-1, k), \quad r = j-1.$$

This gives the recurrence relation:

$$P_0(n, k) = q \sum_{r=1}^j p^{r-1} P_0(n-r, k) \\ + p^{j-1} \{P_0(n-j+1, k-1) - q \sum_{r=1}^{j-1} p^{r-1} P_0(n-r-j+1, k-1)\}.$$

From this is derived the generating function of the process:

$$U(s, t) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} P_0(n, k) t^n s^k \\ = \frac{p^j t^j (1-pt)(1-p^j t^{j-1})}{(1-t+qp^j t^{j+1})^2} s \\ \times \sum_{k=1}^{\infty} \left\{ \frac{p^{j-1} t^{j-1} (1-t+qp^j t^{j+1})}{1-t+qp^j t^{j+1}} \right\}^{k-1} s^{k-1},$$

where  $0 \leq i \leq j-1$ .

Extensions are indicated to the case in which the process is already under way at  $t=1$ , and towards finding the expected number of trials required for the  $k$ th occurrence of  $E$ . Physical applications are mentioned.

{The formulas above for  $P_0(n, k)$  and  $U(s, t)$  are corrections to (13) and (25) of the paper, communicated by the author; also, just before (17), for  $P_0(n, 0)$  read  $P_0(n, 0)$ .}

P. Masani (Bloomington, Ind.)

6002:

Daboni, Luciano. Una proprietà delle distribuzioni poissoniane. Boll. Un. Mat. Ital. (3) 14 (1959), 318-320. (English summary)

The author gives a proof of a theorem proposed by J. Lamperti [Amer. Math. Monthly 66 (1959), 317]. The theorem is to the effect that in a Bernoullian scheme with a random number  $N$  of trials the number  $N_A$  of successes and the number  $N_B = N - N_A$  are independent if and only if  $N$  has a Poisson distribution. The author deduces that in this case each of  $N_A$  and  $N_B$  has a Poisson distribution. His proof of the theorem depends on a functional equation for the probability generating function of  $N$ .

H. P. Mulholland (Exeter)

6003:

Ishii, Keiiti. Bounds on probability for non-negative random variables. Ann. Inst. Statist. Math. Tokyo 11 (1959), 89-99.

Let  $\mu_0, \mu_1, \dots, \mu_n$  be real numbers, and let  $P = P(\mu_0, \dots, \mu_n)$  be the class of all non-negative,  $\sigma$ -additive measures on  $R = [0, \infty)$  with  $r$ th moment  $\mu_r$  ( $r = 0, \dots, n$ ). Let  $E$  be a given closed (or open) set contained in  $R$ . The author gives sharp upper and lower bounds for  $P(E)$ , as  $P$  ranges over  $P(\mu_0, \dots, \mu_n)$ . His results extend those of A. Wald [Trans. Amer. Math. Soc. 46 (1939), 280-306; MR 1, 14].

E. Parzen (Stanford, Calif.)

6004:

Jitina, Miloslav. On regular conditional probabilities. Czechoslovak Math. J. 9 (84) (1959), 445-451. (Russian summary)

Let  $X$  be an abstract space,  $S$  be a  $\sigma$ -algebra of its subsets, and  $\pi$  be a probability measure on  $S$ . Let  $\pi(\cdot, A)$  be the not necessarily uniquely determined conditional probability of  $A \in S$  relative to a  $\sigma$ -algebra  $T \subset S$ . Only measurability relative to  $T$  completed by adjunction of subsets of  $T$ -null sets is demanded. The class of all functions  $\pi(\cdot, \cdot)$  is denoted by  $\Pi(T)$ , and  $\Pi(T)$  is called regular if there exists a  $\pi$  for which  $\pi(x, \cdot)$  is a probability measure for each  $x$ , or semiregular if only finite additivity is required. The following theorems are proved. (A) If  $T$  has a countable basis,  $\Pi(T)$  is semiregular. (B) Under certain assumptions, for example if  $X$  is a locally compact Hausdorff space and if  $S$  is generated by the compact  $G_\delta$  sets, semiregularity of  $\Pi(T)$  implies regularity. From (A) and (B) it follows that: (C) if the hypotheses of (B) hold and if  $T$  has a countable basis, then  $\Pi(T)$  is regular. The author states in a note added in proof that he now can prove (A) and hence (C) without the hypothesis of a countable basis for  $T$ . These results go beyond some previously obtained by the author [same J. 4 (79) (1954), 372-380; MR 16, 1034].

J. L. Doob (Urbana, Ill.)

6005:

Rényi, A. On measures of dependence. Acta Math. Acad. Sci. Hungar. 10 (1959), 441-451. (Russian summary, unbound insert)

A measure of dependence  $\delta(x, y)$  of two random variables  $x, y$  on a probability space should satisfy seven postulates: (A)  $\delta(\xi, \eta)$  is defined for any pair of random variables  $\xi$  and  $\eta$ , neither of them being constant with probability 1. (B)  $\delta(\xi, \eta) = \delta(\eta, \xi)$ . (C)  $0 \leq \delta(\xi, \eta) \leq 1$ . (D)  $\delta(\xi, \eta) = 0$  if and only if  $\xi$  and  $\eta$  are independent. (E)  $\delta(\xi, \eta) = 1$  if there is a strict dependence between  $\xi$  and  $\eta$ , i.e., either  $\xi = g(\eta)$  or  $\eta = f(\xi)$ , where  $g(x)$  and  $f(x)$  are Borel-measurable functions. (F) If the Borel-measurable functions  $f(x)$  and  $g(x)$  map the real axis in a one-to-one way onto itself,  $\delta(f(\xi), g(\eta)) = \delta(\xi, \eta)$ . (G) If the joint distribution of  $\xi$  and  $\eta$  is normal, then  $\delta(\xi, \eta) = |R(\xi, \eta)|$ , where  $R(\xi, \eta)$  is the correlation coefficient of  $\xi$  and  $\eta$ .

In the light of these postulates, the author examines a number of commonly used indices of dependence: the correlation coefficient  $R(\cdot, \cdot)$ , the correlation ratios  $\theta_x(y) = \sup_{f \in \mathcal{F}} R(f(x), y)$  and  $\theta_y(x)$ , the maximal correlation  $S(x, y) = \sup_{f, g} R(f(x), g(y))$ , the mean square contingency, and a measure due to E. H. Linfoot [Information and Control 1 (1957), 85-89; MR 19, 1148] based on the common entropy  $I(x, y)$ . Only  $S(x, y)$  satisfies postulates (A) to (G). (The information-theoretic measure, contrary to the author's assertion, does not satisfy (E) since for many  $x$   $I(x, x) < \infty$ .)

To calculate  $S(x, y)$ , it is useful to know if there exist  $f_0, g_0$  such that  $S(x, y) = R(f_0(x), g_0(y))$ . Let  $Af = E(E(f(x)|y)|x)$  be an operator on  $L_x^2$  to  $L_x^2$ , the space of random variables that are square integrable functions of  $x$ . Theorem 1: If the transformation  $A$  is completely continuous, then the maximal correlation of  $x$  and  $y$  is attained for  $f_0(x)$  and  $g_0(y)$ , where  $f_0$  is an eigenfunction belonging to the greatest eigenvalue  $S^2 = S^2(x, y)$  of  $A$  and  $g_0(y) = (1/S)E(f_0(x)|y)$ . The dependence between  $x, y$  is called regular if their joint distribution is absolutely

continuous with respect to the direct product of their distributions. Theorem 2: If the dependence between  $x$  and  $y$  is regular and the mean square contingency is finite, then the transformation  $A$  is completely continuous and thus the maximal correlation of  $x$  and  $y$  can be attained.

H. P. Kramer (Santa Barbara, Calif.)

6006:

Rényi, A. On the central limit theorem for the sum of a random number of independent random variables. *Acta Math. Acad. Sci. Hungar.* **11** (1960), 97-102. (Russian summary, unbound insert)

The author generalizes an earlier result [same *Acta* **8** (1957), 193-199; MR **19**, 467] that is a corollary of one due to Anscombe [*Proc. Cambridge Philos. Soc.* **48** (1952), 600-607; MR **14**, 487]; namely, that if

$$Y_{N_n} = N_n^{-1/2} \sum_{i=1}^{N_n} X_i$$

is a sequence of positive integer-valued random variables, where the  $X_i$  are independent and identically distributed random variables with zero expectation and unit variance such that  $n^{-1}N_n$  converges in probability to a positive constant, then  $\lim_n P(Y_{N_n} < x) = \Phi(x)$ . His generalization is the theorem: If  $N_n$  ( $n=1, 2, \dots$ ) is such a sequence of positive integer-valued random variables that  $n^{-1}N_n$  converges in probability to a positive random variable  $Z$  having a discrete distribution, then the sequence  $Y_{N_n}$  converges in probability to a Gaussian random variable. He further shows by counterexample that the weaker supposition that  $n^{-1}N_n$  converges in distribution to a positive random variable  $Z$  having a discrete distribution is not sufficient.

H. P. Edmundson (Pacific Palisades, Calif.)

6007:

Pakshirajan, R. P. On the maximum partial sums of sequences of independent random variables. *Teor. Veroyatnost. i Primenen.* **4** (1959), 398-404. (Russian summary)

In a paper by the reviewer [*Trans. Amer. Math. Soc.* **64** (1948), 205-233; MR **10**, 132] a strong limit theorem which forms a complement to the law of the iterated logarithm is proved in terms of maximum partial sums  $\max_{1 \leq r \leq n} |S_r|$ . Each random variable is assumed to have a finite third absolute moment and an order condition is imposed on the ratio of this to the second moment (the first being 0). This condition is now weakened by use of Berry's estimates involving truncated second moments instead of Esseen's involving third moments. Let  $F_r$  be the distribution of the  $r$ th variable and  $s_n$  the standard deviation of the  $n$ th partial sum; the new condition reads:  $s_n \rightarrow \infty$  and

$$\frac{1}{(\log s_n)^{1/2} s_n^{3-2\alpha}} \sum_{r=1}^n \int_{|x| > s_n^{1-\alpha}} x^2 dF_r(x) = O(1)$$

for some  $\alpha$  in  $(0, 3/4]$ . If all  $F_r$  are the same and the absolute moment of order  $2+\delta$ ,  $\delta > 0$ , is finite then the new condition is satisfied, so that the strong limit theorem is valid. The reviewer wonders if this remains valid when only the second moment is finite, as is the case for the law of the iterated logarithm and was proved by Hartman and Wintner [*Amer. J. Math.* **63** (1941), 169-176; MR **2**, 228].

K. L. Chung (Syracuse, N.Y.)

6008:

Badrikian, Albert. Convergence de la répartition empirique vers la répartition théorique. *C. R. Acad. Sci. Paris* **250** (1960), 1789.

Résumé de l'auteur: "L'objet de cette Note est d'étudier une forme de convergence de la répartition empirique vers la répartition théorique pour des variables aléatoires à valeurs dans un espace topologique localement compact dénombrable à l'infini. On généralise un résultat de Varadarajan [*Sankhyā* **19** (1958), 23-26; MR **20** #1348]."

6009:

Kimme, Ernest G. Some equivalence conditions for the uniform convergence in distribution of sequences of stochastic processes. *Trans. Amer. Math. Soc.* **95** (1960), 495-515.

Let  $(\Omega, \mathcal{B}, p)$  be a probability space and let  $x_n(t, \omega)$ ,  $t \in T$ , be a sequence ( $n=1, 2, \dots$ ) of real stochastic processes defined on it. In same *Trans.* **84** (1957), 208-229 [MR **18**, 770] the author investigated the convergence as  $n \rightarrow \infty$  of such a sequence when the processes had independent increments, and he gave a convergence criterion that insured the convergence in distribution of the sequence of random vectors

$$(1) \{ \sup [x_n(t, \omega), t_{j-1} < t \leq t_j], \inf [x_n(t, \omega), t_{j-1} < t \leq t_j] \}$$

for the case  $t_j = j/N$ ,  $1 \leq j \leq N$ ,  $T = [0, 1]$ ,  $t_0 = 0$ ,  $n \geq 1$ . Here, the author gives several equivalent minimal conditions under which the sequence (1) converges, and then concludes that the hypothesis of uniform convergence in distribution is of much greater significance to the general problem of convergence on process sequences than was indicated in his earlier work.

H. P. Edmundson (Pacific Palisades, Calif.)

6010:

Lamperti, John. On null-recurrent Markov chains. *Canad. J. Math.* **12** (1960), 278-288.

Let  $P$  be the transition matrix of an irreducible, null-recurrent Markov chain in discrete time. Derman [*Proc. Amer. Math. Soc.* **5** (1954), 332-334; MR **15**, 722] showed that there is a positive vector  $Q$ , unique except for a scalar factor, such that  $Q = QP$ . The first question studied is: given a sequence of vectors  $\{U^{(n)}, n \geq 0\}$  such that  $U^{(n)} = U^{(0)}P^n$ , what can one say about the convergence of  $U^{(n)}$  to  $Q$ ? It is assumed that  $U^{(0)}$  is real and is bounded by a scalar multiple of  $Q$  so that the multiplication is assured. It is shown that Abel-summability of (each component of)  $\{U^{(n)}\}$  implies that its Abel-limit is  $Q$ . On the other hand, given any bounded but not convergent sequence  $\{x_i\}$ , there is a permutation  $\{x_i'\}$  of it such that if  $U_i^{(0)} = x_i'Q$ , then  $\{U^{(n)}\}$  is not Abel-summable. Positive criteria are also given for two special types of chains. Next, the following random scheme is considered. Let  $N_i^{(0)}$  particles be at state  $i$  initially and let all particles move independently of each other according to the transition matrix  $P$ , and let  $N_i^{(n)}$  be the number of particles at state  $i$  at time  $n$ . Derman [*Trans. Amer. Math. Soc.* **79** (1955), 541-555; MR **17**, 50] showed that if for each  $i$   $N_i^{(0)}$  is Poisson with mean  $Q_i$ , then the vector process  $\{N^{(n)}, n \geq 0\}$  is stationary. Here a sufficient condition, rather similar to that for sums of independent random variables to converge to a Poisson limit distribution, is

given so that the above vector process will converge to Derman's stationary one. Another scheme, where particles are fed continually to a fixed state, is considered.

K. L. Chung (Syracuse, N.Y.)

6011:

Siraždinov, S. H. A local limit theorem for a Markov chain with continuous time. *Izv. Akad. Nauk UzSSR. Ser. Fiz.-Mat.* 1958, no. 6, 83-86. (Russian. Uzbek summary)

Consider a homogeneous Markov chain with a finite number of states  $e_1, \dots, e_s$  and continuous time parameter. Let  $\xi^a(t)$  be the duration of staying in state  $e_a$  for a time  $t$  and let  $\xi(t) = (\xi^1(t), \dots, \xi^s(t))$ . Let  $\eta(t) = t^{-1/2}[\xi(t) - t\omega] = (\eta^1(t), \dots, \eta^s(t))$ , where  $\omega = (\omega_1, \dots, \omega_s)$ . Further, let  $W_r(t, X)$  be the density function corresponding to the absolutely continuous component of the probability function  $P[\eta(t) \in E | e(0) = e_r]$  and let  $W_r(t, x)$  be the density function corresponding to the absolutely continuous component of the distribution function

$$P\{\sigma^{-1/2}[U_t - t(h\omega)] < x | e(0) = e_r\}.$$

The author derives two expressions: one for  $W_r(t, X)$  in terms of the multivariate Gaussian density plus an error term and similarly a second one for  $W_r(t, x)$ . These improve the author's earlier results in *Dokl. Akad. Nauk SSSR* 98 (1954), 905-908 [MR 16, 494].

H. P. Edmundson (Pacific Palisades, Calif.)

6012:

Harders, Hartwig. Über stationäre Markovprozesse mit abzählbar vielen Zuständen, insbesondere elementare Geburtsprozesse. *Math. Nachr.* 17, 156-187 (1959).

In this paper, based on the author's Göttingen dissertation, certain problems are considered for stationary Markov processes with a countable number of states, in particular, elementary birth processes. The paper is divided into four sections: § 1, Stationary Markov processes with a countable number of states; § 2, Birth processes; § 3, Elementary birth processes; and § 4, Quasi-regular elementary birth processes. Let  $\mathcal{M} = \{X(t), t \geq 0\}$  be a stationary Markov process with a countable number of states, and transition probabilities  $P_{mn}(t)$ .  $\mathcal{M}$  is said to be an elementary birth process if (i)  $P_{mn}(t) = 0$  for  $n < m$ , and (ii)  $P_{mn}'(0) = 0$  for  $n > m + 1$ .

In § 1, after introducing terminology and notation, and stating results of Austin, Chung, Doob, Kolmogorov, and Lévy, the author proves the following theorem (Satz 1.4): Let  $m$  be fixed. (1) The following three assertions are equivalent: (i)  $D \sum_{n=1}^{\infty} P_{mn}(t) = \sum_{n=1}^{\infty} DP_{mn}(t)$  is valid for  $t=0$ ; (ii)  $DP_{mn}(t+s) = \sum_{i=1}^{\infty} [DP_{mi}(t)]P_{in}(s)$  is valid for  $t=0$  and  $n=1, 2, \dots$ ; (iii)  $\sum_{n=1}^{\infty} DP_{mn}(t+s) = \sum_{i=1}^{\infty} [DP_{mi}(t)] \times \sum_{n=1}^{\infty} P_{in}(s)$  is valid for  $t=0$ . (2) The following three assertions are equivalent: (i)  $P_{mn}'(t) = -q_m P_{mn}(t) + \sum_{i=1}^{\infty} q_{mi} P_{in}(t)$  (the backward Kolmogorov equation) is valid for  $n=1, 2, \dots$ ; (ii)  $D \sum_{n=1}^{\infty} P_{mn}(t) = -q_m \sum_{n=1}^{\infty} P_{mn}(t) + \sum_{i=1}^{\infty} q_{mi} \sum_{n=1}^{\infty} P_{in}(t)$ ; (iii) (ii) is valid for  $t=0$ . In the above,  $D$  denotes differentiation with respect to  $t$ . The relations stated under (1) were established for  $t > 0$  by D. G. Austin [*Proc. Nat. Acad. Sci. U.S.A.* 41 (1955), 224-226; MR 16, 1130].

A state  $m$  is called regular if  $\sum_{n=1}^{\infty} P_{mn}(t) = 1$ , irregular otherwise. The process  $\mathcal{M}$  is termed regular if this condition is satisfied for all  $m$ . If the condition  $\sum_{n=1}^{\infty} P_{mn}'(0) = 0$  is satisfied the state  $m$  is called quasi-regular. Let  $p_{mn} = \delta_{mn}$  for  $q_m = 0$ , and  $q_{mn}/q_m$  otherwise. Here, as above,

the  $q_{mn}$  are the infinitesimal transition probabilities. Then, a necessary and sufficient condition that a state  $m$  be quasi-regular is that  $\sum_{n=1}^{\infty} P_{mn} = 1$ . Theorem (Satz 1.5): If only a finite number of states are accessible from  $m$ , then  $m$  is regular if and only if all these states are quasi-regular. (A state  $n$  is said to be accessible from a state  $m$  if  $P_{mn}(t) > 0$ .)

Let  $E_m(t; f) = \sum_{n=1}^{\infty} f(n) P_{mn}(t)$ , where  $f(n)$  is any (complex-valued) number-theoretic function, denote the generalized moments of the process  $\mathcal{M}$ . Since  $P_{mn}(0) = \delta_{mn}$ ,  $E_m(0; f) = f(m)$ . Theorem (Satz 1.6): There exists a number  $\tau_m(f)$  such that  $E_m(t; |f|)$  is finite for  $0 \leq t \leq \tau_m(f)$  and infinite for  $t > \tau_m(f)$  ( $0 \leq \tau_m(f) \leq \infty$ ), and the following hold: (i)  $\tau_m(f) \geq \tau_m(g)$  when  $n$  is accessible from  $m$ , and (ii)  $E_m(s+t; f) = \sum P_{mn}(s) E_n(t; f)$  for  $0 \leq s+t \leq \tau_m(f)$ , the summation extending over the states  $n$  which are accessible from  $m$ .

The main result in § 2 is the following theorem (Satz 2.1): Among semimatrices  $P = (P_{mn}(t))$ ,  $n \geq m$ , of complex-valued functions  $P_{mn}(t)$ , continuous for  $t \geq 0$ , there exists for a given set of complex numbers  $q_m, q_{mn}$  ( $n > m$ ), a unique solution of the system of functional equations  $P_{mn}(s+t) = \sum_{i=m}^{\infty} P_{mi}(s) P_{in}(t)$ ,  $m \leq n$ , satisfying the initial conditions  $\lim_{t \rightarrow 0} (1 - P_{mm}(t))/t = q_m$  and  $\lim_{t \rightarrow 0} P_{mn}(t)/t = q_{mn}$  ( $m < n$ ). This solution is the solution of both the forward equation  $P_{mn}'(t) = \sum_{i=m}^{\infty} P_{mi}(t) q_{in} - P_{mn}(t) q_n$  for  $n \geq m$ , and the backward equation  $P_{mn}'(t) = -q_m P_{mn}(t) + \sum_{i=m+1}^{\infty} q_{mi} P_{in}(t)$  for  $n \geq m$ , satisfying the initial condition  $P_{mn}(0) = \delta_{mn}$ . A corollary of the Feller-Dobrushin theorem is proved, a statement of which is as follows: A necessary condition for the regularity of a birth process is the divergence of  $\sum_{n=m}^{\infty} q_n^{-1}$  for all  $m$ . A property of  $E_m(t; f)$  for a birth process is given by the following theorem (Satz 2.4): Let  $m$  be a regular state of a birth process; and let  $f(n)$  be a monotone nondecreasing number-theoretic function. Let  $f(N) > f(m)$  for a given state  $N$  which is accessible from  $m$ . Then  $E_m(s+t; f) > E_m(s; f)$  for  $s+t < \tau_m(f)$ .

In § 3 several representations of the transition probabilities  $P_{mn}(t)$  for an elementary process are given; and conditions are stated which an elementary process must satisfy in order that it be regular. The main result in this section is a proof of the following theorem (Satz 3.4): For every state  $n$ , with  $q_n > 0$ , which is accessible from a state  $m$  ( $m < n$ ), there exists a  $t_{mn} > 0$  such that  $P_{mn}'(t) > 0$  for  $0 < t < t_{mn}$ ,  $= 0$  for  $t = t_{mn}$  and  $P_{mn}''(t_{mn}) < 0$ , and  $< 0$  for  $t > t_{mn}$ . Furthermore,  $t_{mn} < t_{m, n+1}$  and  $t_{mn} > t_{m+1, n}$ . Now let  $T_m = \lim_{n \rightarrow \infty} t_{mn}$ . This limit (finite or infinite) will exist when  $p_n > 0$  for  $n \geq m$  (here  $p_n = q_{n, n+1}$ ). Theorem (Satz 3.5): A state  $m$  of an elementary process is regular only if  $T_m = \infty$ , and all states which are accessible from  $m$  are quasi-regular. From this result it follows that an elementary process is regular only if  $p_n = q_n$  and  $T_n = \infty$  for all  $n$ .

In § 4 the author considers quasi-regular elementary processes, all of whose states are irregular. In this case  $0 < p_n = q_n$ , and  $\sum_{n=1}^{\infty} q_n^{-1} < \infty$ . It is also assumed that  $q_m \neq q_n$  for  $m \neq n$ . Let  $P_n(t) \equiv P_{1n}(t)$ . It is then shown that

$$\sum_{n=1}^{\infty} P_n(t) = \sum_{n=1}^{\infty} e^{-q_n t} \prod_{r=1}^n q_r (q_r - q_n)^{-1},$$

where the accent on the product sign indicates omission of all meaningless factors. The Dirichlet series

$$D(t) = \sum_{n=1}^{\infty} e^{-q_n t} \prod_{r=1}^{\infty} q_r (q_r - q_n)^{-1}$$

is introduced, and the following theorem proved (Satz 4.1): When, for a given  $t \geq 0$ ,  $D(t)$  is absolutely convergent, then  $\sum_{n=1}^{\infty} P_n(s)$  converges uniformly for  $s \geq t$ , and  $\sum_{n=1}^{\infty} P_n(s) = D(s)$ . A. T. Bharucha-Reid (Eugene, Ore.)

6013:

Yuškevič, A. A. On strong Markov processes. Teor. Veroyatnost. i Primenen. 2 (1957), 187-213. (Russian. English summary)

The article gives the following conditions under which a trajectory of a Markov process  $x(t)$  is right-continuous. Let  $X$  be a  $\sigma$ -compact metric space and  $\mathfrak{B}$  the  $\sigma$ -algebra generated by the open sets of  $X$ , and assume  $X$  admits the introduction of a metric  $\rho(x, y)$ , a  $\mathfrak{B} \times \mathfrak{B}$ -measurable function, with respect to which  $X$  is complete. If for the transition function  $p(s, x, t, \Gamma)$  ( $\Gamma \in \mathfrak{B}$ ,  $x \in X$ )

$$\lim_{t \downarrow s} p(u, x, t, V_\varepsilon(x)) = 0 \quad (s \leq u \leq t)$$

for every  $s \geq 0$  and  $\varepsilon > 0$  uniformly in  $x$ , where  $V_\varepsilon(x) = \{y \in X, \rho(x, y) \leq \varepsilon\}$ , then there exists a Markov process with trajectories right-continuous in the metric  $\rho$ , having  $p(s, x, t, \Gamma)$  as its transition function.

The author gives a condition for a process with trajectories right-continuous to be strongly Markovian, generalizing the condition given by E. B. Dynkin [Dokl. Akad. Nauk SSSR 113 (1957), 261-263; MR 20 #7348], and studies in detail strong Markov processes with trajectories right-continuous in the discrete topology.

A measurable Markov process with trajectories not defined for all  $t > 0$ , and with measurable transition function, is called an 'almost strong Markov process' by the author if for any  $s \geq 0$ , for any variable  $\tau(\omega)$  independent of the future and of the  $s$ -past, with domain of definition  $\Omega_\tau$ , and for any variable  $\eta(\omega)$  ( $\omega \in \Omega_\tau$ ) determined by the behavior of the process on the interval  $[s, \tau]$  and such that  $\eta \geq \tau$ , one has

$$P_{s,x}(\Omega_\tau, x(\eta) \in \Gamma) = \int_{\Omega_\tau} p(\tau, x(\tau), \eta, \Gamma) dP_{s,x} \quad (\Gamma \in \mathfrak{B}).$$

The author shows by an example that not every measurable transition function is the transition function of a strong Markov process. However, in the countable case, for any  $p_s(t)$  which is the transition function of a homogeneous process right-continuous at zero, he constructs an almost strong Markov process with transition function equal to  $p_s(t)$ . M. G. Šur (RŽMat 1958 #7969)

6014:

Šur, M. G. On the Feller property of Markov processes. Dokl. Akad. Nauk SSSR 129 (1959), 1250-1253. (Russian)

The author considers continuous parameter Markov processes with stationary transition probabilities. The state space is locally compact and separable, the sample functions right-continuous. Let  $\{T_t\}$  be the usual associated semigroup from the space  $L$  of bounded Borel-measurable functions into itself. A  $C_0$ -neighborhood of a point is a set including the point with the property that almost all sample paths from the point remain in the neighborhood for a strictly positive time. The process is said to be of  $C_0$ -Feller type if  $T_t$  takes the class of  $C_0$ -continuous functions in  $L$  into itself. It is proved that

every strong Markov process is of  $C_0$ -Feller type. If  $M$  is the class of  $C_0$ -continuous functions in  $L$  which are right-continuous on almost every path from each point, each of the following conditions (to hold for all positive  $s$ ) is necessary and sufficient that the process be strongly Markov: if  $f$  is continuous and bounded,  $T_s f \in M$ ;  $T_s M \subset M$ . J. L. Doob (Urbana, Ill.)

6015:

Lévy, Paul. Processus strictement ou presque strictement markoviens. Compositio Math. 14, 172-193 (1959).

The author discusses his titular subject, stressing the usefulness of defining a random function of a parameter by constructive methods (obtaining it successively at the points of a parameter sequence and then using a separability argument for the remaining parameter values). The analytical details are not always given in full.

J. L. Doob (Urbana, Ill.)

6016:

Meyer, André. Fonctions de transition subordonnées. C. R. Acad. Sci. Paris 250 (1960), 1962-1964.

Let  $P_t(x, A)$ ,  $t > 0$ ,  $x \in X$ ,  $A$  a measurable subset of  $X$ , be the conditional probability that a process at "time"  $s+t$  has its value in  $A$ , given that the value is  $x$  at "time"  $s$ , and suppose  $P$  is the transition function of a stationary Markov process. If  $Q$  has the same domain as  $P$ ,  $Q \leq P$ , and  $Q$  satisfies all the conditions of a Markov process except that of total transition probability one, then  $Q$  is a transition function subordinate to  $P$ . The author's principal result is that, essentially, a  $Q$  sample function is a  $P$  sample function which disappears at some random time. Under suitable separability assumptions, various strong Markov properties are obtained. No proofs are given.

H. Rubin (E. Lansing, Mich.)

6017:

Matveev, R. F. Regularity of multidimensional stationary random processes with discrete time. Dokl. Akad. Nauk SSSR 126 (1959), 713-715. (Russian)

It is proved that an  $n$ -dimensional stationary process (discrete parameter) is regular of rank  $m$  if and only if the following conditions are satisfied. (I) The spectral distribution matrix function is absolutely continuous. (II) The rank of the corresponding density matrix function  $\|f_{ij}(\lambda)\|$  is  $m$  almost everywhere. (III) There is a principal minor  $M(\lambda)$  of the density matrix, of order  $m$ , with

$$\int_{-\pi}^{\pi} \log \det M(\lambda) d\lambda > -\infty.$$

From now on suppose that  $M(\lambda) = \|f_{ij}(\lambda)\|$  ( $i, j \leq m$ ) and denote by  $M_{ik}(\lambda)$  the determinant of the matrix obtained from  $M(\lambda)$  when the  $k$ th row in the full density matrix is replaced by the  $i$ th row. (IV) The functions  $M_{ik}/M$  ( $i = m+1, \dots, n$ ;  $k = 1, \dots, n$ ) are the boundary functions (radial limits) of meromorphic functions of class  $N_\delta$  for some  $\delta > 0$ . {Note by reviewer: The author calls a meromorphic function of class  $N_\delta$  one which is the quotient of two regular functions on the unit disc, each of Hardy class  $H_\delta$ . The class  $N_\delta$  is then, however, independent of  $\delta$ , and is the class of functions of bounded characteristic.} The author bases his work on a set of necessary and sufficient conditions obtained by Rozanov [Uspehi Mat. Nauk 13 (1958), no. 2 (80), 93-142; MR 22 #5076].

J. L. Doob (Urbana, Ill.)

6018:

Rozanov, Yu. A. On the extrapolation of generalized stationary random processes. *Teor. Veroyatnost. i Primenen.* **4** (1959), 465-471. (Russian. English summary)

A generalized stationary random process is a continuous, translation-invariant, linear map  $\xi$  from  $D$ , the space of infinitely differentiable functions  $\phi$  with compact support on the real line, to a Hilbert space of random variables. [See K. Ito, *Mem. Coll. Sci. Univ. Kyoto Ser. A Math.* **28** (1954), 209-223; MR **16**, 378; I. M. Gel'fand, *Dokl. Akad. Nauk SSSR* **100** (1955), 853-856; MR **16**, 938]. Let  $H_s^-$  be the closed manifold spanned by the  $\xi(\phi)$  with  $\phi(t)=0$  for  $t \geq s$ . The process is called singular if all  $H_s^-$  are the same, regular if  $\bigcap_s H_s^- = (0)$ . The author characterizes these classes by conditions on the associated spectral measure. He also solves the prediction problem, that is, he finds a formula for the element of  $H_s^-$  closest to  $\xi(\phi)$ , with  $\phi$  in  $D$ . The results extend the well-known ones for ordinary second-order stationary processes.

G. A. Hunt (Princeton, N.J.)

6019:

Nizio, Makiko. On polynomial approximation for strictly stationary processes. *J. Math. Soc. Japan* **12** (1960), 207-226.

Given a Brownian motion with increments  $B(I, \omega)$  ( $I=[a, b] \subset R^1$ ) and shift transformation  $B(I, \omega_s^+) = B([a+s, b+s], \omega)$ , N. Wiener [*Amer. J. Math.* **60** (1938), 897-936] proved that the homogeneous polynomials

$$p_n(t, \omega) = B(I_1, \omega_t^+) B(I_2, \omega_t^+) \cdots B(I_n, \omega_t^+),$$

( $I_1, I_2, \dots, I_n \subset R^1, n \geq 1$ ) span the class of ergodic stationary processes in a suitable topology. Wiener's argument was hard to follow, and the author has performed a valuable service in publishing this new and elegant proof.

Given a Brownian path, if  $a_n(\omega) - n =$

$$\inf\{\theta: \theta > -n, |B([-1, +1], \omega_s^+)| > 1, \theta - n < s \leq \theta\},$$

then  $\lim_{n \rightarrow \infty} [a_n(\omega_t^+) - a_n(\omega)] = -t$ , and, roughly speaking,  $a_n(\omega)$  becomes uniformly distributed on  $[0, +\infty)$  as  $n \rightarrow +\infty$ ; thus, singling out an individual sample path  $x$  of the ergodic stationary process  $\mathfrak{P}$  which is typical in the sense that its evolution mirrors the ensemble, the composite process  $\mathfrak{P}_n: x_n(t) \equiv x(-a_n(\omega_t^+))$  will approximate  $\mathfrak{P}$  as  $n \rightarrow +\infty$  and can be approximated, in turn, by Hermite polynomials of the Brownian increments  $B(I, \omega_t^+)$  according to the theorem of Cameron, Martin, and Wiener [see R. H. Cameron and W. T. Martin, *Ann. of Math.* (2) **48** (1947), 385-392; MR **8**, 523].

The restriction to ergodic processes is unnecessary, as the author shows by another argument.

H. P. McKean, Jr. (Cambridge, Mass.)

6020:

Gorman, C. D. Brownian motion of rotation. *Trans. Amer. Math. Soc.* **94** (1960), 103-117.

Brownian motion on the surface of a 3-sphere is constructed, by a direct application of the formulas for the composition of probabilities, from the plane Brownian motion. Thus the construction is entirely different from the analytical approaches by F. Perrin [*Ann. Sci. École Norm. Sup.* (3) **45** (1928), 1-51] and the reviewer [*Ann. Math. Statist.* **20** (1949), 292-296; MR **10**, 721]. The construction is described as follows. Let  $(x(t), y(t))$ ,

$0 \leq t < \infty$ , be the plane Brownian motion. For each non-negative integer  $n$ , let the interval  $[0, 1]$  be subdivided by the points  $k/2^n$  ( $k=0, 1, \dots, 2^n$ ). Let us put

$$x^{(n)}(t) = x\left(\frac{k-1}{2^n}\right) + 2^n\left(t - \frac{k-1}{2^n}\right)\left[x\left(\frac{k}{2^n}\right) - x\left(\frac{k-1}{2^n}\right)\right],$$

$$y^{(n)}(t) = y\left(\frac{k-1}{2^n}\right) + 2^n\left(t - \frac{k-1}{2^n}\right)\left[y\left(\frac{k}{2^n}\right) - y\left(\frac{k-1}{2^n}\right)\right]$$

for  $(k-1)/2^n \leq t < k/2^n$  ( $k=1, \dots, 2^n; n=0, 1, \dots$ ). Let  $S$  be a sphere of radius 1 placed on the  $x$ - $y$  plane touching at the origin. Assume that  $S$  rolls without slipping along the polygonal path  $\{(x, y) | x=x^{(n)}(t), y=y^{(n)}(t), 0 \leq t \leq 1\}$  in such a way that it has constant angular velocity along each linear portion of the path corresponding to the intervals of time  $(k-1)/2^n \leq t < k/2^n$  ( $k=1, \dots, 2^n$ ). Let, for each  $t \in [0, 1]$ ,  $R^{(n)}(t)$  be the rotation of  $S$  around its centre as it rolls in the above manner from the origin to the point  $(x^{(n)}(t), y^{(n)}(t))$ . Thus, for each sample path of the plane Brownian motion process, we obtain a sequence  $R^{(n)}(t)$  ( $0 \leq t \leq 1; n=0, 1, \dots$ ) of sample paths in the space of the group of rotations of the 3-sphere. The author's main theorem states that  $\lim_{n \rightarrow \infty} R^{(n)}(t)$  exists uniformly in  $t$  ( $0 \leq t \leq 1$ ), for almost all sample paths of the plane Brownian motion process. K. Yosida (Tokyo)

6021:

Lévy, Paul. Deux nouvelles extensions du mouvement brownien. *C. R. Acad. Sci. Paris* **250** (1960), 966-968.

In the first part of this paper the author makes some intuitive remarks on Brownian motion in a Hilbert space. In the second part he considers a Gaussian stochastic process  $\{X(A), A \in \Sigma\}$ , where  $\Sigma$  is the surface of a sphere in  $N$  dimensions. It is supposed that  $E\{X(A)\} = 0$  and that  $E\{X(A)X(B)\}$  is the surface measure of the intersection of the hemispheres centered at  $A$  and  $B$ . Then  $X(A) - X(B)$  has variance proportional to the distance on  $\Sigma$  between  $A$  and  $B$ , so that the  $X(A)$  stochastic process is Brownian motion with parameter set  $\Sigma$ . This approach is the analogue of Čencov's approach [*Dokl. Akad. Nauk SSSR* **106** (1956), 607-609; MR **17**, 1101] to Brownian motion with plane parameter set. Details will be given elsewhere.

J. L. Doob (Urbana, Ill.)

6022:

Gupta, H. C. Diffusion in the presence of barriers. *J. Math. and Phys.* **39** (1960/61), 58-63.

The author considers an unsymmetric random walk of a particle on a straight line, with continuous time, and with both absorbing and reflecting barriers. The velocity of the particle at any time, if still not absorbed, is either  $u_1 > 0$  or  $u_2 < 0$ , where  $u_1$  and  $u_2$  are constant, and the infinitesimal probability of transition from one velocity to another is constant. Formulae are obtained for the probability that the particle is, at time  $t$ , in any assigned infinitesimal interval and is moving to the left, or to the right.

I. J. Good (Teddington)

6023:

Harris, T. E. A lower bound for the critical probability in a certain percolation process. *Proc. Cambridge Philos. Soc.* **56** (1960), 13-20.

The author considers the "percolation process" on a

square lattice as originally defined by Broadbent and Hammersley [Proc. Cambridge Philos. Soc. **53** (1957), 629-645; MR **19**, 989]. Links in the lattice are considered active or passive with probability  $p$  or  $1-p$  respectively and one wishes to determine the largest  $p$  for which there is probability  $R(p)=0$  that some given vertex will be part of an infinite set of connected active links. The author proves here that this critical probability is greater than  $\frac{1}{2}$  by showing that  $R(\frac{1}{2})=0$ . The proof makes use of the symmetry between active and passive links for  $p=\frac{1}{2}$  and the symmetry between the lattice and its dual. Roughly speaking the argument is that if  $R(\frac{1}{2})$  were greater than zero one can prove that there would exist, with probability one, active chains enclosing the origin. This in turn implies that one can find no infinite set of connected inactive links on the dual lattice, which leads to a contradiction of the condition that  $R(\frac{1}{2})>0$  must hold also on the dual lattice.

G. Newell (Providence, R.I.)

6024:

Kac, Mark; Slepian, David. Large excursions of Gaussian processes. Ann. Math. Statist. **30** (1959), 1215-1228.

D. S. Palmer [Proc. Cambridge Philos. Soc. **52** (1956), 672-686; MR **18**, 241] and S. O. Rice [Bell System Tech. J. **37** (1958), 581-635; MR **20** #788] studied the distribution of spacings between consecutive  $a$ -values of an ergodic Gaussian process  $x(t)$  ( $Ex(t)=0$ ,  $Ex^2(t)=1$ ). Their procedure is limited to processes for which  $x''(t)$  exists, excluding, for example, the displacement of a harmonic oscillator in Brownian motion. As the process is non-Markovian the above problem requires careful definition of conditional densities, and different modes of definitions, yielding different results, are illustrated. The authors consider  $x(t)$  with covariance function  $\rho(\tau)=1-\frac{1}{2}\alpha\tau^2+o(\tau^2)$ ,  $\alpha>0$ ,  $\tau(0)=1$  (this implies that  $x(0)$  and  $x'(0)$  are independent). Letting  $\theta=\theta(a)$  denote the expected length of the time intervals during which  $x(t)\geq a$  in the sense of Rice, the  $n$ -variate joint limiting distributions of the process  $\Delta(t, \theta)=\theta^{-1}[x(\theta t)-a]$ , as  $a\rightarrow\infty$ , or equivalently as  $\theta\rightarrow 0$ , are obtained under different modes of conditioning. These results require only the existence of the first derivative of  $x(t)$ . The limiting distribution of the first return time to a positive level is studied under slightly stronger assumptions on  $\rho(\tau)$ . The salient feature of both types of theorems is that the limit laws obtained are independent of  $\rho(\tau)$ .

F. L. Spitzer (Minneapolis, Minn.)

6025:

Smith, W. L. Remarks on the paper 'Regenerative stochastic processes'. Proc. Roy. Soc. London. Ser. A **256** (1960), 496-501.

In the author's 1955 paper referred to in the above title [same Proc. **232** (1955), 6-31; MR **17**, 502] the proof of Theorem 1 is incorrect and a correct proof is given here. Moreover, a regularity condition for semi-Markov processes (S.M.P.) on page 21 of the former paper, involving the divergence of  $\sum e_i\eta_i$ , is corrected by redefining the  $e_i$ 's to be truncated above at 1. Additional criteria for regularity of S.M.P.'s are also discussed. For an irregular S.M.P. (one having a positive probability of an infinite number of transitions in a finite interval of time) the author extends the domain of definition to the whole real line by allowing instantaneous reentry from " $\infty$ " in

accordance with some fixed probability distribution over the states. Under this extension, theorem 5 of the original paper provides non-trivial limiting results.

R. Pyke (Seattle, Wash.)

6026:

Khintchine, A. Y. ★Mathematical methods in the theory of queueing. Translated by D. M. Andrews and M. H. Quenouille. Griffin's Statistical Monographs & Courses, No. 7. Hafner Publishing Co., New York, 1960. 120 pp. \$5.50.

Translation of Trudy Mat. Inst. Steklov. **49** (1955) [MR **17**, 276].

6027:

Takács, Lajos. Transient behavior of single-server queueing processes with recurrent input and exponentially distributed service times. Operations Res. **8** (1960), 231-245.

The author considers the queueing system  $GI/M/1$ . Let the successive customers arrive at the epochs  $\tau_0(=0)$ ,  $\tau_1, \tau_2, \dots$ ; let  $x(t)$  denote the number of customers waiting or being served at the epoch  $t$ , and let a transition  $E_k \rightarrow E_{k+1}$  be said to occur at the epoch  $\tau_n$  when  $x(\tau_n-0)=k$ . The author calculates the double generating function with coefficient  $\text{pr}\{x(\tau_n-0)=k | x(0)=i+1\}$  for the term in  $z^k w^n$ , and also the function which has as the coefficient of  $z^k$  in its power series expansion the Laplace transform of  $\text{pr}\{x(t)=k | x(0)=i\}$ . He obtains similar formulae involving the probability  $G_n(X)$  that a busy period consists of  $n$  services and has length at most  $X$ , and the number of transitions  $E_k \rightarrow E_{k+1}$  occurring in the time-interval  $(0, t)$ .

D. G. Kendall (Oxford)

6028:

Finch, P. D. Deterministic customer impatience in the queueing system  $GI/M/1$ . Biometrika **47** (1960), 45-52.

The author (generalising the work of D. Y. Barrer, Operations Res. **5** (1957), 644-656; MR **19**, 779) considers the queueing system  $GI/M/1$  with the following modification; a customer will wait for service in the queue only for a time not exceeding  $W$  (this quantity  $W$  is fixed and is the same for all customers). If he has waited for a time  $W$  and has not been served, he then departs and is permanently lost to the system. The author reduces the problem to the discussion of a certain kind of random walk with two impenetrable barriers, and shows that the integral equation for the equilibrium behaviour of the waiting-time  $w$  has the solution

$$\text{pr}\{w \leq x\} = 1 - \frac{\mu M(W-x) - M'(W-x)}{\mu M(W)} \quad (0 < x < W),$$

where  $M(\cdot)$  has Laplace transform

$$\int_0^\infty e^{-px} M(x) dx = \frac{p - \mu + \mu[a(p) - a(\mu)]}{(p - \mu)[\mu a(p) + p - \mu]},$$

here  $\mu$  is the reciprocal mean service time and  $a(p) = \int_0^\infty e^{-px} dA(x)$ , where  $A(\cdot)$  is the distribution function for the time between consecutive arrivals.

The paper continues with an account of the relation of the above solution to the known solutions (for  $W=\infty$ ) for  $GI/M/1$  and  $M/G/1$ ; the author also discusses the

limiting distribution of queue size, and the extension of his results to the case of  $GI/M/s$ .

(The author promises to publish a corrigendum to deal with an ellipsis in the argument.) *D. G. Kendall* (Oxford)

6029:

Finch, P. D. The output processes of the queueing system  $M/G/1$ . *J. Roy. Statist. Soc. Ser. B* **21** (1959), 375-380.

The equilibrium output of a queue with Poisson arrivals and exponential service times was studied by Burke [Operations Res. **4** (1956), 699-704; MR **18**, 707] and Reich [Ann. Math. Statist. **28** (1957), 768-773; MR **19**, 1203] who showed, in particular, that the sequence of departure instants formed a Poisson process. The author shows that this result is best possible in a certain sense, if arbitrary service time distributions  $B(x)$  are considered, and also if the size of the waiting room is limited. More precisely, it is proved that for a queueing system characterized by (i) a waiting room of size  $N$  ( $0 < N \leq \infty$ ); (ii) a Poisson input process with parameter  $\lambda$ ; (iii) continuous  $B^*(x)$ ,  $B(0+) = 0$ ,  $\int_0^\infty x dB(x) < \infty$ ; and (iv) if  $N = \infty$ , then  $\lambda \int_0^\infty x dB(x) < 1$ ; the following hold if and only if  $B(x)$  is negative exponential: (a) The queue size left by a departing customer is independent, in the limit, of the period since the previous departure. (b) Two successive departure intervals are independent, in the limit.

*E. Reich* (Minneapolis, Minn.)

6030:

Ben-Israel, A.; Naor, P. A problem of delayed service. I, II. *J. Roy. Statist. Soc. Ser. B* **22** (1960), 245-269, 270-276.

(I) A repairman tends a number of machines (liable to break down) and some of his time is taken up by ancillary duties. It is assumed that the attendance times are independent of the state of the machine—working or idle. The authors compare (a) successional as against simultaneous attendance to machines, (b) Poisson breakdown of the whole system as against Poisson breakdown of individual machines, and (c) various assumptions regarding attendance time distributions, in relation to (1) the number of idle machines as observed by a random controller, (2) the number of idle machines as observed by the repairman, and (3) the waiting time of an idle machine.

(II) The authors generalise the above by considering a system in which machines are attended to in groups; machines in the same group are repaired simultaneously while different groups are dealt with in succession.

*D. G. Kendall* (Oxford)

6031:

Miller, H. D. Inter-plant storage in continuous manufacturing. *Technometrics* **2** (1960), 393-401.

The author applies the reviewer's theory of dam storage to a manufacturing problem in which a continuously operating plant  $A$  feeds a product into storage for use by a continuously operating plant  $B$ . Scheduled and unscheduled (random) shutdowns are assumed to occur, and the effects of variations in the size of the store on the total production is examined by setting up the resulting Markov chain. A detailed and interesting numerical example concerning a chemical plant is given.

*P. A. P. Moran* (Oxford)

6032:

Ventura, E. Sur l'utilisation des intégrales de contour dans les problèmes de stocks et de délais d'attente. *Management. Sci.* **6** (1959/60), 423-443.

This paper (presenting results obtained jointly by the author and his staff) commences with the remark that the moments and distribution function of the random variable  $y^+ = \max(y, 0)$ , where  $y$  is a primary random variable, can conveniently be studied by taking the expectation of contour-integrals like

$$(2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} e^{yp} p^{-n} dp, \quad (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} e^{p(k-y)} dp/p;$$

as an illustration the equilibrium formulae for the queueing system  $M/G/1$  are then worked out along these lines.

The paper concludes with a study (by the above method) of the following problem. A mine works regularly for 10 months of the year, but production ceases during the remaining 2 months (say for reasons of climate). Throughout the year ships (all of the same size) arrive at Poissonian intervals to remove the products of the mine. The authors are interested in two sorts of breakdown: (1) a ship will arrive and have to wait for its load; (2) the space assigned for storage may be insufficient. *D. G. Kendall* (Oxford)

#### STATISTICS

See also A5556, 5995, 5996, 6064, 6065, 6092, 6093, 6095, 6098, 6635, 6638, 6646.

6033:

Bauer, Edward L. ★A statistical manual for chemists. Academic Press, New York-London, 1960. x+156 pp. \$4.75.

As the name indicates, this book is intended to present to chemists a cook-book approach for applying the methods and principles of statistics. Such an aim is laudable, and can be brilliantly achieved, as Sir Ronald Fisher's *Statistical methods for research workers* [Oliver and Boyd, Edinburgh, 1925] shows.

The chapter headings in the book are: Fundamentals; The average; Experimental design and the analysis of variance; The comparison of two averages; The comparison of more than two averages; Correlated variables; Sampling; The control of routine analyses. Since these eight chapters take up only 135 pages of text, the explanation is very limited, and it is doubtful whether a chemist could employ the book profitably (on the other hand, one wonders why a topic such as the average deviation, which is never used in the book, is introduced at all). Furthermore, in many places the discussion tends to be inaccurate; in others, irrelevant. One serious drawback in the book is that many terms and symbols are introduced without being defined, and there is a lack of clarity in distinguishing between population and sample. The confusion between sample variance  $s^2$  and population variance  $\sigma^2$  on pages 4 and 8-9 is such that no interpretation of the symbols involved can make the statements meaningful. As a further instance, in discussing the  $t$ -test, the following sentences occur. "If we wish to know if  $\bar{X}_1$  is greater or less than  $\bar{X}_2$  we use only one tail of the  $t$  curve. This is called a one-tailed test. On the other hand, if we wish to know if  $\bar{X}_1$  is part of the population of which  $\bar{X}_2$  is the average, we make use of the whole  $t$  distribution,

and we use a two-tailed test." One might also question the fact that the whole book is based upon a rejection of the use of the standard deviation in favour of an estimate of the standard deviation computed from the range. The author refers to this technique as a "popular" one, but I should prefer to use the adjective "inefficient".

R. G. Stanton (Waterloo, Ont.)

6034:

Fréchet, M. Sur les tableaux dont les marges et des bornes sont données. *Rev. Inst. Internat. Statist.* 28 (1960), 10-32. (English summary)

The problem in question is: Given the values of the marginals  $N_i = \sum_j n_{ij}$ ,  $N'_j = \sum_i n_{ij}$ ,  $1 \leq i \leq q$ ,  $1 \leq j \leq r$ , provide integral entries  $n_{ij}$  of a rectangular table subject to the known bounds  $0 \leq n_{ij} \leq m_{ij}$ . Necessary and sufficient conditions for the existence of a solution are known but the labor involved is evidently prohibitive for  $(q, r)$  outside a small integral neighborhood of  $(2, 2)$ . The author gives an alternative approach to the case  $q=r=3$ , paying special attention to the issue of uniqueness of a solution.

H. Teicher (New York)

6035:

Adam, A. Verteilungsmasse und Verteilungsindizes. *Metrika* 2 (1959), 239-244. (English summary)

Author's summary: "The measures and indexes of distribution discussed in this paper are results of research in the manner of the continental school; they can be interpreted especially in the sense of information theory. Problems of the variance as a statistical aggregate are pointed out, and it is recognized as a special form of the more general measure of distribution. Relations between Bayes' rule and results of regression analysis are briefly discussed."

S. Kullback (Washington, D.C.)

6036:

Raja Rao, B. A formula for the curvature of the likelihood surface of a sample drawn from a distribution admitting sufficient statistics. *Biometrika* 47 (1960), 203-207.

Author's summary: "In this paper is given a formula for the curvature of the 'likelihood surface' (see § 3) of a sample drawn from a distribution admitting sufficient statistics for the parameters, at the point represented by the maximum likelihood estimates (m.l.e.) of the parameters. More precisely it is shown in the case of two parameters that the negative of the trace and the determinant of Fisher's information matrix, respectively, measure the first and second (Gaussian) curvatures of the 'likelihood surface' at the point represented by the m.l.e.'s of the parameters. In the case of  $m$  parameters an expression for the Riemann curvature of the 'likelihood hypersurface' (see § 4) of a sample at the point represented by the m.l.e.'s is obtained in terms of the elements of Fisher's information matrix. An expression for Riemann's curvature invariant is also obtained. A large sample interpretation is given."

M. Loève (Berkeley, Calif.)

6037:

Barankin, Edward W.; Katz, Melvin, Jr. Sufficient statistics of minimal dimension. *Sankhyā* 21 (1959), 217-246.

1928

Let  $p(x, \theta)$ ,  $x = (x_1, \dots, x_n)$ ,  $\theta = (\theta_1, \dots, \theta_r)$ , be a density whose carrier does not depend on  $\theta$ , with continuous partial derivatives  $\partial p / \partial x_i$ ,  $\partial p / \partial \theta_j$ ,  $\partial^2 p / \partial x_i \partial \theta_j$ ,  $\partial^2 p / \partial \theta_j \partial x_i$ . The authors deal with local and global phases of the problem of finding the smallest number of continuously differentiable, real-valued functions which can constitute a sufficient statistic. Principal results: they give integer lower bounds, which are generally exact minima, for the local and the global dimension of a (locally Euclidean, continuously differentiable) sufficient statistic. In fact, they construct two sufficient statistics with local dimension equal to the local bound and the global bound respectively, except possibly on the union of a set of measure 0 and a nowhere dense, closed set. These statistics are also locally functionally minimal in the same region.

J. W. Pratt (Cambridge, Mass.)

6038:

Chapman, D. G.; Robson, D. S. The analysis of a catch curve. *Biometrics* 16 (1960), 354-368.

The author gives the minimum variance unbiased estimate of the annual survival rate  $s$ ,  $\hat{s} = \bar{X}[1 + \bar{X} - (1/n)]^{-1}$ ,  $\bar{X}$  being the mean age of a random sample of size  $n$ , in the case where the age frequency distribution in the population is given by the function  $f(x) = as^x$ ,  $x = 0, 1, 2, \dots$ ,  $a = 1 - s$ . The unbiased estimate of the variance of  $\hat{s}$  is shown to be  $\hat{s}[\hat{s} - (T-1)/(n+T-2)]$  where  $T = n\bar{X}$ . The distribution of  $n\bar{X} = T$  is  $g(t) = \binom{n+t-1}{t} a^n s^t$ ,  $t = 0, 1, 2, \dots$ . From

the property of  $T$  being a sufficient and complete statistic a uniformly minimum variance unbiased estimate of  $s$  exists and the estimate is a unique function  $h(T)$  of  $T$ , if  $s$  is estimable. So, by definition, the expectation of  $h(T)$  is

$$(1-s)^n \sum_{t=0}^{\infty} h(t) \binom{n+t-1}{t} s^t = s;$$

hence we obtain  $\hat{s} = T/(n+T-1)$  as  $h(T)$ . The author obtains the estimate  $\hat{s}$  by the maximum likelihood method from the truncated sample, because it is impossible to construct an unbiased estimate of  $s$ . Other methods of estimation of  $s$ , say, Jackson's method, the regression estimate and the maximum likelihood method based directly on the definition of  $s$  are discussed under various assumptions. Also the author gives the procedure for testing the validity of  $f(x) = as^x$  with particular emphasis on the zero class.

C. Hayashi (Tokyo)

6039:

Singh, Naunihal. Estimation of parameters of a multivariate normal population from truncated and censored samples. *J. Roy. Statist. Soc. Ser. B* 22 (1960), 307-311.

Suppose  $k$  random variables are distributed independently, the distribution of each being a truncated normal distribution. The original, untruncated distribution is the same for each variable, but the interval of truncation varies from variable to variable. This paper derives maximum likelihood estimation procedures for the first two moments of the original untruncated distribution, and a study is made of the asymptotic behavior of the covariance matrix of some of these estimates.

M. Dwass (Evanston, Ill.)

6040:

Cornish, E. A. Fiducial limits for parameters in compound hypotheses. *Austral. J. Statist.* 2 (1960), 32-40.

Let  $x_1, \dots, x_N$  be  $N$  independent random variables having normal distributions with means  $\xi_1, \dots, \xi_N$  and variances all equal to  $\sigma^2$ . Let  $y_i = \sum_{j=1}^N h_{ij} x_j$ ,  $i = 1, \dots, p$ ,  $p < N$ , be  $p$  linearly independent functions of  $x_1, \dots, x_N$  and let  $s^2$  be an unbiased estimator for  $\sigma^2$  having the chi-square distribution with  $n$  degrees of freedom, such that  $s^2$  is independent of  $x_1, \dots, x_N$  and hence of  $y_1, \dots, y_p$ . Let  $t_i = (y_i - \alpha_i)/sd_i$ ,  $i = 1, \dots, p$ , where  $d_i = [\sum_j h_{ij}^2]^{1/2}$ , and  $\alpha_i = \sum_j h_{ij} \xi_j$ . Let  $r_{ij} = \sum_{k=1}^N h_{ik} h_{jk} / d_i d_j$ , and  $\|r^{ij}\| = \|r_{ij}\|^{-1}$ . It has been shown by Cornish [*Austral. J. Physics* 7 (1954), 531-542; MR 16, 602] and by Dunnett and Sobel [*Biometrika* 41 (1954), 153-169; MR 15, 885] that the probability element of  $(t_1, \dots, t_p)$  is given by

$$(1) \frac{\Gamma[(n+p)/2]}{\sqrt{(|r_{ij}|)(\pi n)^{p/2} \Gamma(n/2)}} [1 + \sum_{i,j=1}^p r^{ij} t_i t_j]^{-(n+p)/2} dt_1 \dots dt_p.$$

Using this probability element and the  $t$ 's, that is,

$$\frac{y_1 - \alpha_1}{sd_1}, \dots, \frac{y_p - \alpha_p}{sd_p},$$

as pivotal quantities, the author observes that the fiducial probability element of  $\alpha_1, \dots, \alpha_p$  is obtained from (1) as

$$(2) \frac{\Gamma[(n+p)/2]}{\sqrt{(|a_{ij}|)(\pi n)^{p/2} \Gamma(n/2)}} \times [1 + \sum_{i,j=1}^p a^{ij} (y_i - \alpha_i)(y_j - \alpha_j)]^{-(n+p)/2} d\alpha_1 \dots d\alpha_p,$$

where  $a_{ij} = s^2 \sum_{k=1}^N h_{ik} h_{jk}$ , and  $\|a^{ij}\| = \|a_{ij}\|^{-1}$ . The author discusses several special cases of (2) and some illustrative numerical examples. S. S. Wilks (Princeton, N.J.)

6041:

Dhondt, André. Sur une généralisation d'un théorème de R. Frisch en analyse de la confluence. *Cahiers Centre Études Rech. Oper.* 2, 37-46 (1960).

More direct proof of the generalization by Reiersøl [see *Ark. Mat. Astr. Fys.* 32A, 1945, no. 4; MR 7, 317] of a theorem by Frisch [see *Statistical confluence analysis by means of complete regression systems*, Univ. Økonomiske Inst., Oslo, 1934] concerning the coefficient vector of a linear structural relation involving  $n$  variables.

G. E. Noether (Boston, Mass.)

6042:

Konijn, H. S. Some remarks on regression analysis with uncontrolled regressors. *Austral. J. Statist.* 2 (1960), 47-52.

This paper considers the use of regression analysis when the regression variables are not under experimental control (a) in discovering essential relationships versus (b) the prediction of the values of some variable. Discussion leads to the question of the possible exclusion of a regressor  $Z_k$  under unchanged structure. Denote the sum of squares of the  $n$  observations on  $Z_k$  from their arithmetic mean by  $s_k^2$  and let  $\beta_k$  be the true partial regression coefficient of  $Z_k$  in the regression of  $Y$  on  $Z_1, \dots, Z_k$ . Let  $\hat{z}_k^0$  be the least squares prediction of  $Z_k$  from  $Z_1, \dots, Z_{k-1}$  when the latter take on the values  $z_1^0, \dots, z_{k-1}^0$ . Let  $R$  be the multiple correlation between  $Z_k$  and  $Z_1, \dots, Z_{k-1}$ . Then the true variance of the prediction  $\hat{Y}^0$ , when  $Z_k$  is excluded,

decreases by the product of the true variance of the disturbances from the true regression and  $(\hat{z}_k^0 - z_k^0)^2 \cdot s_k^{-2} (1 - R^2)^{-2}$ . The bias of the prediction is  $\beta_k (\hat{z}_k^0 - z_k^0)$ . So when  $s_k$  is small or  $R$  is large, one can reduce the variance of prediction  $\hat{Y}^0$  for  $Z_1, \dots, Z_k$  taking on the values  $z_1^0, \dots, z_k^0$  by eliminating  $Z_k$ , provided  $Z_k^0$  is not well predicted for  $Z_1, \dots, Z_{k-1}$  at  $z_1^0, \dots, z_{k-1}^0$ . One should not do this unless the square of the bias is small in comparison with this variance. The proof covers the case of the exclusion of several regressors.

P. S. Dwyer (Ann Arbor, Mich.)

6043:

Sibuya, Masaaki. Bivariate extreme statistics. I. *Ann. Inst. Statist. Math. Tokyo* 11 (1960), 195-210.

Let  $X, Y$  be random variables,  $F(x, y) = \Pr\{X < x, Y < y\}$ ,  $G(x) = \Pr\{X < x\}$ ,  $H(y) = \Pr\{Y < y\}$ . The equation  $F(x, y) = \Omega(G(x), H(y))G(x)H(y)$  defines the dependence function  $\Omega$  of  $(X, Y)$ . Let  $(X_i, Y_i)$  ( $i = 1, \dots, n$ ) be mutually independent, distributed as  $(X, Y)$ , and let  $X_{\max} = \max X_i$ ,  $Y_{\max} = \max Y_i$ . The dependence function  $\Omega_n$  of  $(X_{\max}, Y_{\max})$  satisfies the equation  $\Omega_n(G, H) = \Omega^n(G^{1/n}, H^{1/n})$ . The paper is concerned with the asymptotic behavior of  $\Omega_n$  under the assumption that  $F(x, y)$  is continuous. Necessary and sufficient conditions are given for  $\lim_{n \rightarrow \infty} \Omega_n(G, H) = 1$  (asymptotic independence of  $X_{\max}$  and  $Y_{\max}$ ), for  $\lim_{n \rightarrow \infty} \Omega_n(G, H) = \min(G^{-1}, H^{-1})$  (degenerate asymptotic distribution) and for  $\Omega_n(G^{1/n}, H^{1/n}) = \Omega(G, H)$  ( $n = 1, 2, \dots$ ). W. Hoeffding (Chapel Hill, N.C.)

6044:

Brunk, H. D. Mathematical models for ranking from paired comparisons. *J. Amer. Statist. Assoc.* 55 (1960), 503-520.

There are  $m$  items to be compared in pairs.  $S_{ij}$  is a random variable which measures the score on the comparison of the  $i$ th item with the  $j$ th. It is assumed that  $S_{ij} + S_{ji} = 1$ , the conventional situation being one in which the score has only two possible values, 0 and 1. To say that the  $m$  items are "ranked" means that a permutation,  $R(1), \dots, R(m)$  of the integers  $1, \dots, m$  is assigned. Let  $e_{ij} = ES_{ij}$  and let  $e^{ij} = e_{R(i)R(j)}$ . A "utility of a ranking" is defined to be a function of the scores  $e^{ij}$  which is non-decreasing in its arguments, and an "optimum ranking" is one which maximizes this function. The spirit of the paper is to unify the many special cases discussed in the literature; many specific kinds of utility functions are examined, analogies with the analysis of variance are exploited and certain technical aspects of utility functions are examined mathematically. M. Dwass (Evanston, Ill.)

6045:

Wetherill, G. B. The Wilcoxon test and non-null hypotheses. *J. Roy. Statist. Soc. Ser. B* 22 (1960), 402-418.

Let the two (continuous) populations under consideration have distribution functions  $F(x)$  and  $G(y)$ . The paper investigates the asymptotic behavior of the Wilcoxon test if (i)  $F(x)$  and  $G(y)$  are normal with slightly different variances; (ii)  $F(x)$  is standard normal,  $G(y)$  has small but non-zero skewness and/or kurtosis. The main results may be summarized as follows. The Wilcoxon test is somewhat more robust than the  $t$ -test to differences in variance, but

it is much more sensitive to skewness and kurtosis than the  $t$ -test. Differences of skewness can obscure differences of means or medians which it is desired to test. (In Tables 2 and 3, the headings should read  $\mu$  and  $\sigma$  instead of  $\sigma'$  and  $\mu$ .)

G. E. Noether (Boston, Mass.)

6046:

DeBaun, Robert M.; Chew, Victor. Optimum allocation in regression experiments with two components of error. *Biometrics* 16 (1960), 451-463.

Consider a functional relation  $y = f(x, y, \dots) + \sum e_i$ , where the  $e_i$  are error terms from experimentally distinguishable distributions. The authors have made a commendable start towards determining optimum experimental designs for estimating the parameters of the function where: (1) the number of independent variables is two or less, (2) the relation is linear, (3) fitting is by least squares, (4) error distributions are two or less, (5) errors from different distributions are uncorrelated, (6) errors from one of the distributions are mutually uncorrelated for all  $N$  observations, (7) errors from the other distribution, if it exists, are identical in equal subsets of  $G$  observations and uncorrelated for observations from disjoint subsets, (8) both error distributions are independent of the magnitude of the independent variables, (9) the "cost" of the experiment is a linear function of the number of (distinct) observations on each variate, with a fixed and known ratio between them, and (10) the regression coefficient for each variable is independent of the values of the other variable.

The authors are interested in efficient estimation at a particular set of values of the independent variables, which happens to involve extrapolation, and this guides their development. The exposition, introduction, accuracy of the title, choice of symbols, obiter dicta, and proof-reading do not match the standard of the principal development. In particular, the above list of assumptions, due to the reviewer, may be neither complete nor a faithful reading of the authors' accomplishments.

C. J. Maloney (Frederick, Md.)

6047:

Basman, R. L. An expository note on estimation of simultaneous structural equations. *Biometrics* 16 (1960), 464-480.

The following structural model is considered (as an example, to illustrate the general procedure).  $x_i$  ( $i = 1$  to 4),  $y_1$ ,  $y_2$  are observed quantities;  $y_r$  is considered as an observed value  $y_r = \eta_r + \varepsilon_r$  of a parameter  $\eta_r$ , whose random error  $\varepsilon_r$  has zero expectation. The following relations hold

$$\eta_1 = \beta_{12}\eta_2 + \gamma_{11}x_1 + \gamma_{12}x_2 + \gamma_{13},$$

$$\eta_2 = \beta_{21}\eta_1 + \gamma_{23}x_3 + \gamma_{24}x_4 + \gamma_{25}.$$

The coefficients are to be estimated. Three types of least-square estimation procedure are proposed and discussed, mainly on the basis of empirical results: a full theoretical approach to the problem of their relative efficiencies is still lacking.

C. A. B. Smith (London)

6048:

Wright, Sewall. The treatment of reciprocal interaction, with or without lag, in path analysis. *Biometrics* 16 (1960), 423-445.

The author shows how the method of path coefficients can be adapted to deal with problems of reciprocal interaction, in which character  $A$  influences character  $B$  and vice versa. If there is a time lag, the equations are set up by considering the situation at discrete times separated by the lag, leading to systems of simultaneous equations. Cases of simultaneous interaction are studied by bringing in extra hypothetical variables, as shown by examples, one economic, and one concerned with data on  $\text{CO}_2$  concentration in respiration. C. A. B. Smith (London)

6049:

Pasternack, Bernard S. Analysis of covariance for a  $3 \times 4$  triple rectangular lattice design (3 associate p. b. i. b.). *Biometrics* 16 (1960), 7-18.

An illustration of the necessary calculations for the analysis of covariance for a three associate partially balanced incomplete block design using data from agricultural field trials. The discussion of the analysis is particularly illuminating with respect to the interpretation of the analysis of covariance.

M. Zelen (College Park, Md.)

6050:

Dugué, Daniel. ★Un mélange d'algèbre et de statistique: Le plan d'expériences. Université de Paris. Les Conférences du Palais de la Découverte, Série A, No. 257. Édition du Palais de la Découverte, Paris, 1960. 18 pp. 1.30 NF.

Light expository lecture on orthogonal squares and incomplete balanced blocks.

L. J. Savage (Ann Arbor, Mich.)

6051:

Rao, P. V. On the construction of some partially balanced incomplete block designs with more than three associated classes. *Calcutta Statist. Assoc. Bull.* 9 (1960), 87-92.

Using methods similar to those advanced by Shah to construct PBIB designs with three associate classes [*Ann. Math. Statist.* 30 (1959), 48-54; MR 21 #2343], the author constructs PBIB designs with  $2m+1$  associate classes. He replaces Shah's balanced matrices by partially balanced ones.

R. L. Anderson (Raleigh, N.C.)

6052:

Eberl, W. Statistische Kontrollen in der industriellen Produktion. *Österreich. Ing. Z.* 3 (1960), 96-103.

6053:

Wierzbicki, W. A comparative survey of strength theories from the probabilistic point of view. *Proc. 4th Congress Theoret. Appl. Mech.* 1958, pp. 279-284. *Indian Soc. Theoret. Appl. Mech.*, Kharagpur.

6054:

Kitagawa, Tosio. Successive process of statistical controls. III. *Mem. Fac. Sci. Kyushu Univ. Ser. A* 14 (1960), 1-33.

[For part II, see same *Mem.* 13 (1959), 1-16; MR 21 #3927.] The author indicates how the analysis of certain

statistical methods using control charts can be evaluated using the theory of Markov chains. Most of the paper is devoted to the application of a theorem of Romanowsky and Gantmacher to special Markov transition probability matrices. This theorem expresses the  $n$ th step transition probability matrix in terms of the determinant and cofactors of the characteristic matrix of the first step transition probability matrix.

H. Chernoff (Stanford, Calif.)

6055:

Zelen, Marvin. Factorial experiments in life testing. *Technometrics* 1 (1959), 269-288.

The paper presents methods for analyzing life test data when the data are taken over a factorial array of different environmental conditions. The analysis is given in detail when the underlying distributions of life for each treatment condition are exponential. The techniques can also be applied, after obvious modifications, to the analysis of a factorial array of variance estimates when the underlying distributions are normal [see, e.g., R. E. Bechhofer, *J. Amer. Statist. Assoc.* 55 (1960), 245-264; MR 22 #4099]. The application of the usual analysis of variance formulae to the logarithms of the observed failure times may be reasonably robust, when the underlying lifetime distributions are not exponential.

Benjamin Epstein (Palo Alto, Calif.)

6056:

Sanchez-Crespo Rodriguez, Jose Luis. Sampling of finite populations: cluster and area sampling. *Estadist. Española* No. 7 (1960), 18-38. (Spanish. English summary)

This expository article, part of a series on the standard theory of survey sampling, deals with clusters of unequal sizes. Its treatment follows the textbook of Hansen, Hurwitz and Madow, but in outline form. It deals mostly with random sampling and subsampling of unequal clusters and touches but slightly on the practical problems of stratified cluster sampling and probabilities proportional to size.

Leslie Kish (Ann Arbor, Mich.)

6057:

Calleja Siero, Anselmo. Sampling of finite populations: sampling clusters of equal sizes, without subsampling. *Estadist. Española* No. 6 (1960), 55-70. (Spanish)

This expository article contains in outline the standard textbook material described by the subtitle; the treatment resembles mostly those of Cochran and Deming. The aim is a series of several articles to present to the readers of the Spanish journal the main ideas of the standard theory of survey sampling.

Leslie Kish (Ann Arbor, Mich.)

6058:

Hald, A. The compound hypergeometric distribution and a system of single sampling inspection plans based on prior distributions and costs. *Technometrics* 2 (1960), 275-340.

In the first part of the paper the author summarizes previous studies on single sampling plans for the acceptance of lots based on attributes. Items are classified either as defectives or non-defectives. Usually four quantities determine a single sampling plan:  $p_1$ ,  $p_2$ ,  $\alpha$ ,  $\beta$ . From a lot

of  $N$  items, a sample of  $n$  items ( $n < N$ ) is drawn with  $c$  defectives,  $c \leq n$ . If the number of defectives in the sample is  $c$  or less, the lot is accepted, otherwise the lot is subjected usually to 100% inspection or disposed of in some other fashion. Here  $p_1$  is the proportion of defectives in a lot acceptable to the consumer;  $\alpha$  is the probability of rejecting such lots (the producer's risk);  $p_2 > p_1$  is the proportion of defectives in a lot which is unacceptable to the consumer, and  $\beta$  is the probability of accepting such lots. From these specifications,  $(n, c)$  is determined. The author is interested in developing criteria for  $(n, c)$  not based on  $(p_1, p_2, \alpha, \beta)$ , but based on the idea of a prior distribution and the economic costs of any sampling plan, specifically  $k_1$  and  $k_2$  as defined subsequently.

By  $\Phi_N(p)$  we denote the cumulative prior distribution, the probability that the fraction defective of a submitted lot of  $N$  items is less than or equal to  $p$ . The average costs per lot submitted for inspection is denoted by  $K(n, c)$ , the cost function. The cost function consists of the costs of sampling inspection, the expected loss due to accepted defects, and the costs of a rejected lot. The problem, to find the optimum plan (the values of  $n$  and  $c$ ) reduces to the minimizing of  $K(n, c)$  for a given prior distribution  $\Phi_N(p)$  and given values  $k_1$ , the costs per item of sampling inspection, and  $k_2$ , the costs per item of rejected lots after sampling inspection. The minimum cost is then compared with  $\bar{p}_N$  and  $k_2$  to find out whether to accept the lot without sampling inspection, to reject without sampling inspection, or to sample the lot. Here  $\bar{p}_N$  is the average cost for acceptance without inspection,  $\bar{p}_N = \int_0^1 p d\Phi_N(p)$ . The author considers two models for cost. He also reviews previous uses of the prior distribution.

In order to solve the problem of minimum costs the author devotes a long section to the compound hypergeometric distribution. Let  $f_N(X)$  denote the probability that a lot of  $N$  items contains  $X$  defective items, and let  $p(x|X)$  denote the probability that a sample of  $n$  items contains  $x$  defectives when the lot contains  $X$  defectives. Then letting  $y = X - x$ ,  $g_n(x)$  is called the compound hypergeometric distribution,

$$g_n(x) = \binom{n}{x} \sum_{y=0}^{N-n} f_N(x+y) \binom{N-n}{y} / \binom{N}{x+y}.$$

Here  $y = X - x$  denotes the number of defective items in the non-inspected part of the lot. He finds the joint first and second moments and central moments of  $(x, y)$ , and  $p_n(x)$ , the average fraction defective in the non-inspected part of the lots, given that the sample defective found in the samples is  $x/n$ .

Special prior distributions considered are the hypergeometric distribution, the binomial distribution, the Polya distribution, a mixed binomial distribution, and the beta distribution. The mixed binomial distribution is defined by

$$f_N(X, p_i, w_i) = \sum_{i=1}^m w_i \binom{N}{X} p_i^X q_i^{N-X}, \quad \sum w_i = 1.$$

The optimum sampling plan corresponding to a given prior distribution, given lot size and cost parameters is found by solving two inequalities for  $n$  and  $c$ . These inequalities are too involved to reproduce here. If more than one solution exists, the solution that makes  $K(n, c)$  a minimum is chosen and a lower limit to the minimum costs is also found. The two inequalities result from the minimization of  $K(n, c)$ , the cost function.

Next it is proved that for prior distributions with a continuous limit and a linear or nearly linear regression function the sample size,  $n$ , increases proportionally to  $N^{1/2}$  for large  $N$ . For prior distributions with a discontinuous limit the sample size increases proportionally to  $\log N$  for large  $N$ .

Tables are given for  $n, c$  and  $N$  assuming  $f_N(X) = 1/(N+1)$ , the rectangular distribution for a prior distribution, with  $k_r = k_s = .25$ . In another example involving a particular prior Polya distribution with  $k_s = k_r = 0.2, 0.3, 0.4$ , the author finds  $n, c$  for given  $N$  for optimum sampling. For a prior mixed binomial  $p_1 = 0.1, p_2 = 0.5, w_1 = 0.8, w_2 = 0.2$ , the author determines  $n, c$  for a given  $N$  assuming the same values of  $k_r$  and  $k_s$  as for the Polya distribution. These results are compared with the corresponding Polya distribution.

The author expects to publish tables of  $n, c$  for given  $N$  for the Polya distribution and for the mixed binomial distribution with 2 components.

L. A. Aroian (Los Angeles, Calif.)

6059:

Wetherill, G. B. Some remarks on the Bayesian solution of the single sample inspection scheme. *Technometrics* 2 (1960), 341-352.

The author gives simplified methods for determination of  $c$ , the acceptance number, in a sample of  $n$  from a lot. It is assumed that the prior distribution of the probability,  $p$ , of a defective item is the mixed binomial distribution  $(a_i, p_i)$  ( $i = 1, 2, \dots, k$ ),  $\sum_{i=1}^k a_i = 1$ , where  $a_i$  is the probability with which a lot has the proportion defective  $p_i$ . The values of  $c$  depend on  $n$ , the loss function, and  $p_i$ . The author states that the method appears to be robust over wide changes of the parameters in the model.

L. A. Aroian (Los Angeles, Calif.)

6060:

Cox, D. R. Serial sampling acceptance schemes derived from Bayes's theorem. *Technometrics* 2 (1960), 353-360.

Consider a sequence of lots  $\{\dots, L_{n-1}, L_n, L_{n+1}, \dots\}$  to be accepted or rejected, and suppose the number of defectives in random samples of fixed size from these lots is given by the sequence  $\{\dots, x_{n-1}, x_n, x_{n+1}, \dots\}$ . Assume that the  $x_n$  have a Poisson distribution with mean  $m_n$ . The stochastic process  $\{m_n\}$  is specified such that  $m_n$  has only two possible values,  $a$  and  $b$ ,  $a < b$ . Lots with  $m_n = a$  are called good and lots with  $m_n = b$  are called bad. The sequence  $\{m_n\}$  is assumed to be a realization of a simple Markov chain with transition matrix

$$\begin{pmatrix} 1-t_a & t_a \\ t_b & 1-t_b \end{pmatrix}.$$

The author determines the optimum rule for rejecting a lot  $L_n$  based on  $x_{n-1}$  and  $x_n$ , a one step back sentencing rule. He then discusses rejection regions for rejecting  $L_n$  based on  $(x_{n-k}, x_{n-k+1}, \dots, x_n)$ , looking backward over  $(k+1)$  previous samples, and applies the same methods to simultaneous backward and forward looking sample such as  $(x_{n-k}, x_{n-k+1}, \dots, x_n, \dots, x_{n+k-1}, x_{n+k})$ . The hypergeometric and binomial distribution may be substituted in principle for the Poisson.

L. A. Aroian (Los Angeles, Calif.)

6061:

Barnard, G. A.; Lindley, D. V.; Hill, B.; Anscombe, F. J.; Good, I. J.; Horsnell, G. Discussion of the papers of Messrs. Hald, Wetherill and Cox. *Technometrics* 2 (1960), 361-372.

G. A. Barnard felt that the system of sampling based on Hald's theory using economic costs and a prior distribution is considerably more useful than the usual plans such as Military Standard 105. B. Hill pointed out that one of the desirable features of any sampling plan is the incentive to the producer to improve his quality. Further, the plan of A. Hald was strongly based on a fixed prior distribution. G. Horsnell referred to some types of deferred sentencing schemes proposed by him. F. J. Anscombe questioned the particular types of cost functions used by A. Hald. He noted the articles of J. V. Breakwell [*J. Amer. Statist. Assoc.* 51 (1956), 243-256], D. Guthrie and M. V. Johns [*Ann. Math. Statist.* 30 (1959), 896-925; MR 22 #1976], and S. Moriguti [*Rep. Statist. Appl. Res. Un. Japan. Sci. Engrs.* 3 (1955), 99-121; MR 19, 1205], all with cost functions different from those of A. Hald. Further, sequential plans are decidedly superior to single sampling acceptance plans, and the asymptotic results of A. Hald and Guthrie and Johns for large  $N$  are less interesting on this account. In his final discussion A. Hald approved the simplified method of G. B. Wetherill for the determination of  $c$  and  $n$ , thought the beta distribution useful as a prior distribution, and did not feel that the specification of a prior distribution would be too difficult. In fact, any sampling scheme assumes some form of a prior distribution.

The answer to a question of I. J. Good as to conditions when one takes  $n \sim \sqrt{N}$  or  $n \sim \log N$ , for large  $N$ , is given precisely in the article already cited of D. Guthrie and M. V. Johns. A. Hald also showed that the cost function of Guthrie and Johns is expressible in terms of the two fundamental parameters  $k_r$  and  $k_s$ . {The reviewer notes that the values of  $(n, c)$  for a given  $N$  depend on a fixed prior distribution. In practice this distribution may have changing parameters. Thus, information must be obtained via sampling in order to estimate these changed parameters and so to shift to a new  $(n, c)$ . Either  $(n, c)$  must be robust or the information gained from the lots as to changing parameters must be extensive enough to determine the parameters of the prior distribution rather accurately. Presumably the same ideas are applicable to sampling by variables. Usually sampling by variables entails a much smaller sample size.}

L. A. Aroian (Los Angeles, Calif.)

## NUMERICAL METHODS

See also 6102.

6062:

Altman, M. Functional equations involving a parameter. *Proc. Amer. Math. Soc.* 11 (1960), 54-61.

Let  $X, M$  be Banach spaces and  $F$  be a real-valued function defined on some neighborhood of  $(x_0, \mu_0)$  in  $X \times M$ . It is required to solve  $F(x, \mu) = 0$ . The technique used is a modification of Newton's method previously introduced by the author [*Bull. Acad. Polon. Sci. Cl. III* 5 (1957), 457-460, 461-465, XXXIX; MR 19, 984]. It is

assumed that  $F$  is majorized by a function  $Q$  of two real variables and that both  $F$  and  $Q$  are twice differentiable and various boundedness and majorizing conditions hold. The convergence of the iteration and its rapidity is established.

R. G. Bartle (Urbana, Ill.)

6063:

Tausky, Olga; Todd, John. Some discrete variable computations. Proc. Sympos. Appl. Math., Vol. 10, pp. 201-209. American Mathematical Society, Providence, R.I., 1960.

(I) Call two sets of points  $A = \{a_1, a_2, \dots, a_m\}$  and  $B = \{b_1, b_2, \dots, b_m\}$  congruent if  $a_i - b_i = \text{constant}$  for  $i = 1, \dots, m$ . We write  $A \simeq_m B$  if  $A$  and  $B$  can be broken up into  $m$  congruent sets. Thus  $A \simeq_3 B$  if  $A = \{0, 1, 11\} + \{4, 5, 18\} + \{8, 14\}$  and  $B = \{0, 1, 11\} + \{7, 8, 21\} + \{4, 10\}$ . The authors seek to know if a  $C$  exists such that  $A \simeq_2 C$  and  $B \simeq_2 C$ . Indeed  $A = \{1, 4, 18\} + \{0, 5, 8, 11, 14\} \simeq_2 C = \{1, 4, 18\} + \{14, 19, 22, 25, 28\} = \{1, 4, 14\} + \{18, 19, 22, 25, 28\} \simeq_2 B = \{8, 11, 21\} + \{0, 1, 4, 7, 10\}$ . This construction is not a priori evident for sets of more than seven points although some progress on the eight-point problem was made by G. A. Dirac and John Todd. The general problem came from Sierpiński [Elem. Math. 5 (1950), 1-4; MR 12, 809]. SEAC gave the above example.

(II) If  $x$  and  $y$  are non-commuting elements such as matrices then  $\log(\exp x \exp y) - x - y$  can be written in terms of commutators by a SEAC program and hand calculations. This is the Baker-Campbell-Hausdorff problem [see K. Goldberg, Duke Math. J. 23 (1956), 13-21; MR 18, 572].

(III) In the British football pool there are three possibilities in betting (on  $n$  games): win  $W$ , lose  $L$ , or draw  $D$ . When  $n = 4$ , a parsimonious bettor can make nine forecasts as follows:  $DDDD, WWWD, WLDW, DWLW, WDLL, LLLD, LWDL, DLWL, LDWW$ . Then for any outcome he has at most one error. [See O. Tausky and J. Todd, Ann. Soc. Polon. Math. 21 (1948), 303-305; MR 11, 7]. If  $(n = 5)$  games are played, a trivial system of 27 bets can be made with at most one error. We cannot do the same with fewer than  $\lceil 3^5/11 \rceil = 22$  forecasts since by changing the outcome of one or no game, we can multiply the forecasts by at most  $2n + 1 = 11$ . The minimal set of forecasts is as yet unknown, numbering between 23 and 27. A connection is suggested with the basis structure of abelian groups, but nothing is mentioned of machines.

(IV) The Legendre-Sophie St. Germain-Vandiver criterion for Fermat's Last Theorem is discussed [see Vandiver, Amer. Math. Monthly 53 (1946), 555-578; MR 8, 313].

(V) Calculations were made on SEAC to check the conjecture of Ankeny, Artin, and Chowla [see Carlitz, Proc. Amer. Math. Soc. 4 (1953), 535-537; MR 15, 104].

(VI) The class structure of the quadratic forms or ideal classes is studied (with an interesting complication caused by Gaussian versus Kronecker-type classes; see G. Pall, Amer. J. Math. 57 (1935), 789-799). The classical question is considered of whether a single cycle takes care of all classes in the 3-class group. The answer is negative as shown by  $D = -3299$  with ideal classes and  $D = -307$  for forms.

H. Cohn (Tucson, Ariz.)

6064:

Clark, Charles E. The utility of statistics of random numbers. Operations Res. 8 (1960), 185-195.

The possible advantages of using sampling plan techniques, such as stratified sampling, in Monte Carlo type calculations has received considerable attention, especially by H. Kahn [Sympos. Monte Carlo Methods (Univ. of Florida, 1954), pp. 146-190, Wiley, New York, 1956; MR 18, 151]. The present paper has a discussion of how the utilization of tables of pseudo-observations from a given distribution which contain (or have induced) a stratification identification for the entries of the table can yield increased efficiencies by utilizing this stratification when performing a Monte Carlo type calculation. The author illustrates how the table of pseudo-observations from an exponential distribution, which he developed jointly with Holz [see following review] and which has an induced stratification identification, can be useful in studying lengths of queues. However, it is not clear to the reviewer how this particular type of stratification which can be assigned to entries in a table of pseudo-observations can be employed effectively in a computer calculation which generates pseudo-observations as they are needed, i.e., large tables of these generated pseudo observations are not stored. The paper concludes with a brief discussion of one way of introducing a fixed initial queue length into a Monte Carlo type calculation of the distribution of a queue length.

M. Muller (New York)

6065:

Clark, Charles E.; Holz, Betty Weber. ★Exponentially distributed random numbers. Published for Operations Research Office, The Johns Hopkins University by The Johns Hopkins Press, Baltimore, Md., 1960. vii + 249 pp. Paperbound: \$6.50.

The authors present a table of 100,000 pseudo random numbers from the exponential distribution (probability density function  $e^{-x}$ ,  $0 \leq x < \infty$ ) which they generated on a digital computer. The entries are given to four decimal places. The table consists of 200 pages where the entries are listed in the order in which they were generated in blocks of 100 numbers, five blocks per page. Associated with each block of 100 numbers are five statistics which have been derived for the 100 numbers: (1) a chi-square goodness-of-fit value; (2) another related chi-square goodness-of-fit value which intuitively tests for serial dependence; (3) the sample mean and its associated quantile value; (4) the sample second moment and its associated expected quantile value; and (5) the number of runs above and below the sample median as well as the probability that a random number of runs less than or equal to the number observed will occur. Eight tables related to tests of randomness of the observations are also given and the authors are satisfied with the test results which they present. An appendix is included which ranks the blocks of 100 numbers according to the five statistics mentioned above. The obvious way to generate pseudo observations from the exponential distribution would be to generate and transform pseudo random numbers. However, the authors have employed a clever rejection technique which H. Kahn attributes to J. von Neumann. This type of rejection technique, and another one which the authors have not employed and which is possibly more efficient, can be found in Butler [Sympos. Monte Carlo

Methods (Univ. of Florida, 1954), pp. 249-264, Wiley, New York, 1956; MR 18, 152]. On those digital computers which do not have relatively fast subroutines for computing logarithms of pseudo numbers, the rejection technique used here may be more efficient with regard to computation time. However, as was done by the authors, it is then necessary to apply statistical test to the "observations" obtained by the rejection procedure.

M. Muller (New York)

6066:

Rotenberg, A. The calculation of toroidal harmonics. Math. Comput. 14 (1960), 274-276.

This note outlines a method of calculating toroidal harmonics, that is, the associated Legendre functions of half integral order,  $P_{n-1/2}^m(x)$  and  $Q_{n-1/2}^m(x)$ . The method suggested is to evaluate  $P_{n-1/2}^m(x)$  and  $Q_{n-1/2}^m(x)$  from their expansions in terms of a  ${}_2F_1(x)$  hypergeometric series, and then to obtain higher values of  $n$  by the use of a recurrence relation in  $n$ .

The difficulties of trying to form an accurate sum of the series  ${}_2F_1(x)$  near  $x=1$  are not sufficiently emphasised but otherwise the method given here is reasonably fool-proof.

L. J. Slater (Cambridge, England)

6067:

Gordon, N. L.; Flasterstein, A. H. A note on a method of computing the Gamma function. J. Assoc. Comput. Mach. 7 (1960), 387-388.

This note gives a formula for the numerical evaluation of  $P(1+x)$ , in the range  $0 \leq x \leq 1$ , involving Bernoulli numbers, together with an interesting graph of the contours of the bounds of constant error. The results are suitable for application to single- and double-length floating point work on an electronic computer.

L. J. Slater (Cambridge, England)

6068:

Franckx, Ed. La méthode de la moyenne de Césaro utilisée comme méthode de relaxation. Les mathématiques de l'ingénieur, pp. 296-301. Mém. Publ. Soc. Sci. Arts Lett. Hainaut, vol. hors série, 1958.

Describes in general terms some known methods of solving a system of linear algebraic equations, states some known criteria for convergence, and suggests the use of Cesàro means where convergence fails.

A. S. Householder (Oak Ridge, Tenn.)

6069:

Ansorge, R. Über die Konvergenz der Iterationsverfahren zur Auflösung linearer Gleichungssysteme im Falle einer singulären Koeffizientenmatrix. Z. Angew. Math. Mech. 40 (1960), 427.

In case  $A$  is singular and  $Ax=v$  has a solution, the iteration  $Bx^{(k)} + Cx^{(k-1)} = v$ , with  $A = B + C$ , converges provided the proper value  $\lambda = 1$  of  $-B^{-1}C$  has only linear elementary divisors, and all other proper values lie inside the unit circle.

A. S. Householder (Oak Ridge, Tenn.)

6070:

Heinrich, Helmut. Ein inverses Eigenwertproblem für endliche Matrizen und seine graphische Lösung für  $n=3$ . Z. Angew. Math. Mech. 40 (1960), 62-64.

Let  $(\mu_1, \dots, \mu_n)$  be a given  $n$ -tuple of numbers, and let

$A$  be a given  $(n, n)$ -matrix. The problem is to determine a real diagonal matrix  $D = (d_i)$  so that the  $\mu_i$  are the eigenvalues of  $AD$ . This problem arises in molecular spectroscopy and in airframe design, and is solved iteratively by Uhlig [same Z. 40 (1960), 123-125; MR 22 #3103]. The present author reduces the problem to a simultaneous algebraic system of higher degree in the  $d_i$ , and gives a graphical solution for  $n=3$ . This involves drawing a conic section depending only on  $A$ , and then interpolating values determined by the  $\mu_i$ .

G. E. Forsythe (Stanford, Calif.)

6071:

Roy, S. N.; Greenberg, B. G.; Sarhan, A. E. Evaluation of determinants, characteristic equations and their roots for a class of patterned matrices. J. Roy. Statist. Soc. Ser. B 22 (1960), 348-359.

Most of the matrices considered are readily resolved into diagonal and triangular factors. {Some of the determinants can be evaluated by immediate application of one of the identities  $\det(I - \sigma uv^T) = 1 - \sigma v^T u$  and, with obvious notation,  $\det(A) = \det(A_{11}) \det(A_{22} - A_{21}A_{11}^{-1}A_{12})$ .

A. S. Householder (Oak Ridge, Tenn.)

6072:

Fiedler, Miroslav; Pták, Vlastimil. An iterative method of computing the eigenvalues and eigenvectors of a symmetric matrix. Časopis Pěst. Mat. 85 (1960), 18-36. (Czech. Russian and English summaries)

Authors' summary: "The method is suitable for matrices sufficiently near the diagonal form; it may be compared to the Newton method of finding the roots of a polynomial. Let  $A$  be the given complex symmetric matrix and  $D$  be the diagonal matrix having the diagonal elements of  $A$  in its diagonal; these elements are supposed to be different from each other. Then there is exactly one anti-symmetric matrix  $S$  such that  $DS - SD = A - D$ . If all eigenvalues of  $S$  have modulus less than one, the matrix  $E + S^2$  is positive definite. If  $W$  is the positive definite square root of  $E + S^2$ , then the matrix  $U = S + W$  is unitary. The matrix  $\varphi(A) = UAU^*$  is then the first approximation to the diagonal matrix similar to  $A$ . In a similar manner the matrices  $\varphi^2(A) = \varphi(\varphi(A))$ ,  $\varphi^3(A) = \dots$ , are constructed. Now let

$$A = (a_{ik}), c(A) = \min_{(i \neq j)} |a_{ii} - a_{jj}|, Q^*(A) = \sum_{(i \neq j)} |a_{ij}|^2.$$

The following theorem is proved: There are two numbers  $\xi = 0.47172$  and  $\rho = 0.24051$  such that if  $A$  is symmetric and  $c(A) > 0$ ,  $\sigma = Q^*(A)^{1/2} c(A)^{-1} \leq \xi$ , then the iterated matrix  $\varphi^k(A)$  exists for each  $k$  and (1) the sequence  $\varphi^k(A)$  tends to a diagonal matrix  $L$ ; (2)  $Q^*(\varphi^k(A)) < Q^*(A) \rho^k \mu^{2k-1}$ , where  $\mu = \sigma/\xi$ ; (3) the unitary matrix  $V = U_1^* U_2^* U_3^* \dots$  exists and  $L = V^* A V$ .—An estimate of the convergence of the infinite product  $V$  may be given. The method can be applied even in more general cases when some of the diagonal elements are nearly equal; the matrix is then divided into blocks in a suitable manner and a theorem similar to the preceding result may be obtained."

H. Schwerdtfeger (Montreal)

6073:

Ortega, J. M. On Sturm sequences for tridiagonal matrices. J. Assoc. Comput. Mach. 7 (1960), 260-263.

The author draws attention to the fact that in Givens'

method of computing the eigenvalues of a real symmetric matrix [W. Givens, *Numerical computation of the characteristic values of a real symmetric matrix*, Rep. ORNL 1574, Oak Ridge, Tenn., 1954; MR 16, 177] the classical theory of the Sturm sequence requires an extension giving signs to zero values in the sequence. A theorem associating suitable signs to zero terms is proved. It is also pointed out that this does not affect the digital realization on a computer.

H. Schwerdtfeger (Montreal)

6074:

Bachmann, Karl-Heinz. Lösung algebraischer Gleichungen nach der Methode des stärksten Abstiegs. Z. Angew. Math. Mech. 40 (1960), 132-135.

The author proposes to solve an algebraic equation  $f(z) = 0$  in the complex domain by the method of steepest descent (applied to  $|f|^2$ ). Special consideration is given to the case where the sequence of iterates  $z_k$  ( $k = 0, 1, 2, \dots$ ) comes "near" to a zero of the derivative: If one step of the method does not yield a sufficient reduction of  $|f|$  (at least 20%), the search for the next iterate  $z_{k+1}$  is made from  $z_k$  in the direction of the steepest descent as well as in the direction orthogonal to it, and of the various trial points the point with smallest  $|f|$  is finally taken as  $z_{k+1}$ . The corresponding algorithm is described in section 4 of the paper; whether it also takes care of possible effects of round-off errors, the author does not say.

H. Rutishauser (Zürich)

6075:

Miller, J. C. P. Numerical quadrature over a rectangular domain in two or more dimensions. II. Quadrature in several dimensions, using special points. Math. Comput. 14 (1960), 130-138.

[For part I, see Math. Comput. 14 (1960), 13-20; MR 22 #1075.] The author gives a few formulas for numerical quadrature over square and cube. These are mostly contained in the paper by Hammer and Stroud in Math. Tables Aids Comput. 12 (1958), 272-280 [MR 21 #970]. That paper and the theory given by Hammer and Wymore in the preceding paper in same Tables 11 (1957), 59-67 [MR 19, 323] imbed in a broader framework the formulas presented here.

P. C. Hammer (Madison, Wis.)

6076:

Miller, J. C. P. Numerical quadrature over a rectangular domain in two or more dimensions. III. Quadrature of a harmonic integrand. Math. Comput. 14 (1960), 240-248.

Specific formulas are given for numerical evaluation of integrals over square and cube under the assumption that the integrand is harmonic. The assumption permits use of fewer lattice points for the polynomial precision achieved than the formulas have without such restriction. Examples are given.

P. C. Hammer (Madison, Wis.)

6077:

Bahvalov, N. S. Approximate computation of multiple integrals. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. 1959, no. 4, 3-18. (Russian)

The Monte Carlo method for approximating the integral

$$\int_0^1 \dots \int_0^1 f(x_1, \dots, x_m) dx_1 \dots dx_m$$

uses the average of the integrand evaluated at randomly selected points in the region of integration. The order of error in such an approximation is  $C_0 N^{-1/2}$ , where  $C_0$  is a constant and  $N$  is the number of points; no assumption is made about the function. The present paper presents several methods similar to the Monte Carlo which give better approximations for smooth functions.

One result is the following. Assume that  $f(x_1, \dots, x_m)$  has period 1 in each of its variables and that it belongs to a certain class  $H(p, A, \lambda)$ . Then for a certain prescribed choice of  $N$  points the error estimate is of the order  $O(N^{-(p+\lambda)/m})$  and the average value of the error estimate is of the order  $O(N^{-(p+\lambda)/m+1/2})$ .

A. H. Stroud (Madison, Wis.)

6078:

Hsu, L. C. A method for finding precise error bounds of numerical integration formulas in higher dimensions. Acta Math. Acad. Sci. Hungar. 11 (1960), 163-171. (Russian summary, unbound insert)

In this paper is presented a formula for

$$\mathcal{E}(L) = \sup_f \left| \iint_D f(x, y) ds - \sum p_{ij} f(x_i, y_j) \right|,$$

where  $L(f) = \sum p_{ij} f(x_i, y_j)$ ,  $D$  is a domain in the plane and  $f$  is any function in a family of functions with fixed absolute bounds for the function and certain partial derivatives. A principal achievement is the result that there exists a function  $f_0$  in the class at which the value in the formula for  $\mathcal{E}(L)$  is achieved and hence no smaller bound will exist for the class. An application to a formula for the square is then given.

P. C. Hammer (Madison, Wis.)

6079:

Kudryavcev, A. L. On the possibility of the application of electronic digital computers to one of the approximate methods for obtaining conformal maps. Prikl. Mat. Meh. 24 (1960), 390-392 (Russian); translated as J. Appl. Math. Mech. 24, 567-571.

In the complex plane consider a simply-connected region  $G$  bounded by a segment of the real axis and a schlicht continuous arc  $C$  in the upper halfplane which connects the endpoints of this segment. The author describes a simple numerical procedure for the approximate calculation of the center  $a$  and radius  $R$  of the largest possible semicircle that can be inscribed in  $G$ . These values can then be used in the numerical evaluation of the conformal mapping  $w - a = z - a + R^2/(z - a)$ . This mapping transforms  $G$  into a region of similar shape but with smaller "height"  $h$ , where  $h$  is the maximum distance of points of  $C$  from the real axis. Therefore iterated applications of this transformation lead to conformal mapping of the complement of  $G$  (with respect to the upper halfplane) onto regions approximating the upper halfplane.

W. C. Rheinboldt (Syracuse, N.Y.)

6080:

Bridgland, T. F., Jr. A note on numerical integrating operators. II. J. Soc. Indust. Appl. Math. 8 (1960), 531-536.

The numerical methods for the solution of ordinary

linear differential equations with constant coefficients developed by the author in Part I of this paper [same J. 6 (1958), 240-256; MR 20 #2842] are here extended to differential equations with variable coefficients. As in the earlier paper the procedures are based on discrete approximations to multiple integrals, derived and formulated with the help of a shifting operator that acts on infinite sequences. It is shown that these methods, when properly applied, entail a discretization error of order  $O(h^2)$ , where  $h$  is the underlying meshlength, just as in the case of constant coefficients. A related procedure, proposed earlier by R. Boxer and S. Thaler [Proc. IRE 44 (1956), 89-101; MR 19, 1050] involves an error of order  $O(h)$  unless the coefficients of the differential equation are constant, as was shown by Wasow [Z. Angew. Math. Phys. 8 (1957), 401-417; MR 19, 1050]. W. Wasow (Madison, Wis.)

6081:

Klabukova, L. S. Approximate method of solution for the problems of Hilbert and Poincaré. *Vychisl. Mat.* 3 (1958), 34-87. (Russian)

The author considers the problem of Poincaré for the equation

$$(*) \quad \Delta w + p(x, y)w_x + q(x, y)w_y = 0,$$

i.e., the problem of solving (\*) in a region  $T$  subject to  $\alpha(s)w_y + \beta(s)w_x = \gamma(s)$  on  $\partial T$ , where  $\alpha^2 + \beta^2 = 1$ . Two procedures for obtaining approximate solutions to this problem are given. In case  $p$  and  $q$  are sufficiently small the author reduces the problem to an equivalent system of first order equations derived by I. N. Vekua [Mat. Sb. (N.S.) 31 (73) (1952), 217-314; MR 15, 230]. A convergent approximation scheme is then obtained by replacing these equations by suitable finite difference equations. In the general case the author reduces the problem to an integral equation (also due to Vekua, op. cit.) whose solution can be approximated by polynomials in the sense of least squares. Using further results of Vekua [Trudy Tbiliss. Mat. Inst. 11 (1942), 109-139; MR 6, 123] on the reduction of the problem of Hilbert to a pair of Dirichlet problems, the author shows that the solution of the problem of Hilbert can be approximated by finite difference methods.

D. G. Aronson (Minneapolis, Minn.)

6082:

Bahvalov, N. S. Numerical solution of the Dirichlet problem for Laplace's equation. *Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him.* 1959, no. 5, 171-195. (Russian)

In order to increase the precision of the numerical solution of many problems one must increase the number of arithmetic operations. It is a difficult problem to determine the minimal number of operations needed to solve a problem to a prescribed degree of precision. Often it is important to have a solution of the simpler problems: (1) What is the rate of growth of the minimal number of operations required to determine the solution for an increase in the precision? (2) Find a method which will achieve this rate of growth. The author obtains some results for the solution of the Dirichlet problem for Laplace's equation.

Let  $G$  be a bounded region, in  $m$ -dimensional space, with boundary  $\Gamma$ . The author discusses the solution of

$\Delta u = 0$  with the boundary condition  $u|_{\Gamma} = \varphi$  and assumes that  $\Gamma$  belongs to a certain class  $L_{k+1}(B, \mu)$  and  $\varphi$  to a certain class  $H(p, A, \lambda)$ . The following is a summary of the author's results. In order to obtain the solution  $u(P)$  with precision  $\varepsilon$ , by numerical integration of  $u(P) = \int_{\Gamma} K(P, \theta) \varphi(\theta) d\theta$  by the methods proposed in the previous paper [6077 above], it is necessary to use  $O(H_\varepsilon)$  points, where  $H_\varepsilon$  is the  $\varepsilon$ -entropy of the class  $H(p, A, \lambda)$ . To solve the problem by a system of difference equations requires  $O(H_\varepsilon^{m/(m-1)} \log H_\varepsilon |\log \varepsilon|)$  operations which would use  $O(H_\varepsilon^{m/(m-1)})$  words of memory in a computer.

A. H. Stroud (Madison, Wis.)

6083:

Vlasova, Z. A. On the method of reduction to ordinary differential equations. *Trudy Mat. Inst. Steklov.* 53 (1959), 16-36. (Russian)

The author studies the Dirichlet and mixed boundary value problem for the elliptic equation

$$Lu \equiv \frac{\partial}{\partial x} \left( A \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( B \frac{\partial u}{\partial y} \right) - Cu = f$$

in various domains. In particular, following L. V. Kantorovič [Izv. Akad. Nauk SSSR. Otd. Mat. Estest. Nauk 5 (1933), 647-652] he constructs approximate solutions of the form

$$u_n(x, y) = \sum_{k=1}^n \chi_k(x, y) f_k(x) + \chi_0(x, y),$$

where the  $\chi_k$  are linearly independent functions. In the case of the Dirichlet problem, for example,  $\chi_0$  takes on the given boundary data while  $\chi_k f_k = 0$  on the boundary for  $k \geq 1$ . The requirement that

$$I[u_n] = \iint \{A(\partial u_n / \partial x)^2 + B(\partial u_n / \partial y)^2 + Cu^2 - 2fu\} dx dy$$

be minimized leads to a system of ordinary differential equations for the  $f_k$ . The author studies these differential equations and shows that under various conditions the sequence  $\{u_n\}$  converges to a solution of the problem.

D. G. Aronson (Minneapolis, Minn.)

6084:

Strešneva, V. A. Auxiliary tables for solution of the Poisson equation by the method of reduction to ordinary differential equations for a polygonal region. *Trudy Mat. Inst. Steklov.* 53 (1959), 267-282. (Russian)

The approximate solution of the Dirichlet problem for the Poisson equation in polygonal regions by the method of L. V. Kantorovič [Izv. Akad. Nauk SSSR Otd. Mat. Estest. Nauk 5 (1933), 647-652] (see preceding review) reduces to the problem of solving a linear system of ordinary differential equations. The author gives tables of the roots of the corresponding characteristic equations and certain auxiliary coefficients useful for numerical computation by this method. A number of examples are included.

D. G. Aronson (Minneapolis, Minn.)

6085:

Rose, Milton E. A method for calculating solutions of parabolic equations with a free boundary. *Math. Comput.* 14 (1960), 249-256.

The author introduces a novel technique for approximating the solution of the Stefan problem

$$\begin{aligned} u_{xx} &= u_t, & 0 < x < x(t), & t > 0, \\ u_x(0, t) &= -1, & t > 0, \\ (1) \quad \dot{x}(t) &= -u_x(x(t), t), & t > 0, \\ u(x, t) &= 0, & x \geq x(t), & t > 0, \\ x(0) &= 0, \end{aligned}$$

where  $x(t)$  represents a free boundary between two phases, say water and ice, of a heat conducting medium. Previous numerical methods have considered explicitly the position of this boundary. The author prepares to convert the system to

$$\begin{aligned} (\rho c)_t + q_x &= 0, & x > 0, & t > 0, \\ q + kT_x &= 0, & x > 0, & t > 0, \\ (2) \quad T &= \begin{cases} \gamma^- e, & e < 0, \\ 0, & 0 \leq e \leq H, \\ \gamma^+(e-H), & e > H, \end{cases} \\ e(x, 0) &= e_0(x), & x > 0, \\ q(0, t) &= g(t), & t > 0, \end{aligned}$$

where physically  $e$  represents the specific internal energy,  $q$  the heat flux,  $T$  the temperature,  $\rho$  the density,  $k$  the thermal conductivity,  $H$  the latent heat of recrystallization, and  $\gamma^-$  and  $\gamma^+$  the specific heats. Note that the boundary no longer appears explicitly. He conjectures that there is a unique weak solution of (2) and that, under the identification  $T(e) = u$ , its solution coincides with the solution of (1). He then conjectures that the weak solution of (2) is the strong limit of the solutions of a simple, explicit difference analogue of (2). A numerical example is presented to support the conjectures, along with some (well labelled) heuristics. The advantage of the method is that it extends to several space variables in a quite straightforward way; certainly the method deserves further consideration.

J. Douglas, Jr. (Houston, Tex.)

6086:

Lowan, Arnold N. On the numerical treatment of heat conduction problems with mixed boundary conditions. *Math. Comput.* 14 (1960), 266-270.

The author considers the numerical solution of  $T_x = k(T_{xx} + T_{yy})$  on a rectangle when the value of  $T$  is specified along a portion of the boundary and the value of its normal derivative on the remaining portion. It is not assumed that each portion consists of all of some set of sides of the rectangle. He proposes to use the simplest explicit difference equation, along with the most obvious replacement of the boundary conditions. He shows that stability occurs if the usual criterion is satisfied.

J. Douglas, Jr. (Houston, Tex.)

6087:

Albasiny, E. L. On the numerical solution of a cylindrical heat-conduction problem. *Quart. J. Mech. Appl. Math.* 13 (1960), 374-384.

The author discusses the numerical solution of the heat equation in cylindrical coordinates by means of the Crank-Nicolson difference equation. An awkward method

of inducing higher order accuracy is also proposed. The stability of the procedure is discussed, and numerical examples are presented. The examples are intended to show that the initial discrepancies in the solution in the neighborhood of a singularity at a corner of the space-time region decay with increasing time.

J. Douglas, Jr. (Houston, Tex.)

6088:

Szabó, János. Über eine Anwendung der Matrizenrechnung zur Näherungslösung von gewissen partiellen Differentialgleichungen der Festigkeitslehre. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* 1 (1956), 623-631 (1957). (Hungarian. Russian and German summaries)

Consider the partial differential equation  $Lf = p$ ,  $L$  of even order in  $\partial/\partial x$  and  $\partial/\partial y$ ; the derivatives of  $f$  of even order are required to vanish on the boundary of a rectangle. The author observes that when derivatives are replaced by centered differences the resulting finite system of linear equations can be solved explicitly in terms of the well known eigenvalues and vectors of the matrix whose diagonal consists of  $-2-s$  and whose super- and sub-diagonal consist of  $1-s$ . The method amounts to a finite Fourier expansion.

P. D. Lax (New York)

6089:

Radok, J. R. M.; Merrill, R. F. Numerical solution of boundary value problems. *Z. Angew. Math. Mech.* 40 (1960), 202-214. (German and Russian summaries)

The authors set up explicit two-level difference approximations to the heat equation  $u_t = u_{xx}$ , using  $2\beta + 1$  points and with a truncation error  $(\Delta t)^{\beta}$ ,  $\beta = 2, 3, 4, 5$ . It is shown how to take boundary conditions of the form  $u_t + cu_x = 0$  into account; in the special cases  $c = 0$  and  $c = \infty$  (the only ones tested) the boundary schemes devised amount to continuing  $u$  by reflection. Numerical tests indicate the following conclusions. (a) If the initial values are compatible with the boundary conditions, then the gain to be expected from going to a more accurate scheme does take place, at least up to  $\beta = 3, 4$ . (b) If the initial values are inconsistent with the boundary conditions, then there is little or no improvement as  $\beta$  is increased; this is also as expected. But it does seem worth while in these cases to start off the calculations with appreciably smaller values of  $\Delta x$  and  $\Delta t$ . This is due no doubt to the smoothing effect of solving the heat equation. (c) The higher order schemes are stable in the following ranges of  $R = \Delta t/(\Delta x)^2$ : for  $\beta = 2$ ,  $R \leq .66$ ; for  $\beta = 3, 4, 5$ ,  $R \leq .8$ . Since for  $\beta = 1$  the stability interval is  $R \leq .5$ , this shows that, somewhat surprisingly, higher accuracy increases stability. It would be worth while to determine algebraically the precise stability regions.

P. D. Lax (New York)

6090:

Wadey, W. G. Floating-point arithmetics. *J. Assoc. Comput. Mach.* 7 (1960), 129-139.

Methods for estimating the rounding errors which may arise in floating point computation are discussed. The results of programs which estimate such errors are described. It is remarked that such estimation could be performed automatically by a suitably designed electronic computer.

C. B. Haselgrove (Manchester)

## COMPUTING MACHINES

See also 6079, 6090, 6118, 6651.

6091:

★Сборник стандартных и типовых программ для БЭСМ [Collection of standard and typical programs for the BESM]. Izdat. Akad. Nauk SSSR, Moscow, 1960. 75 pp. 4.10 r.

The collection contains ten programs by different authors. Except for two control programs, all are computational in nature: evaluation of determinants (by the method of elimination), solution of systems of linear equations by elimination, inversion of matrices (by the method of rank annihilation), quadratic and cubic interpolation by Newton's formula with divided differences. Chebyshev approximation by least squares, integration of systems of ordinary differential equations by the Runge-Kutta process, and an executive program for computation with complex variables. Each article contains a brief mathematical description, a block diagram, and the machine code for the BESM. A collection such as this points up the need for a universal language for the communication of programs that is independent of any particular computer.

R. N. Goss (San Diego, Calif.)

6092:

Fürst, Gerhard. Maschinenverwendung und Automatisierung in der Statistik. Einführende Bemerkungen. Allg. Statist. Arch. 43 (1959), 313-315.

The paper presents the transcription of the opening remarks of a statistical meeting where the four papers reviewed below [6093, 6094, 6095 and 6096] were included. The opening remarks include a justification for selecting data processing as the theme of the statistical meeting.

M. Muller (New York)

6093:

Szameitat, Klaus. Möglichkeiten und Grenzen der Automatisierung in der Statistik. Allg. Statist. Arch. 43 (1959), 316-333. (2 plates)

This expository paper begins by defining automation as "utilization of machines carrying out specific work sequences without human intervention, automatically, according to programs". Note: This definition makes no mention of feedback. The term "automation in statistics" is then used; however, the term "mechanization of some aspects of data handling in a statistical analysis by use of digital computers" would, in the judgment of the reviewer, be a more precise identification of the activity being discussed in the paper. Precise interpretation of terminology here is extremely important in the light of the title of the paper, namely, "Possibilities and limits of automation in statistics". While there might be a problem of language translation, it is felt there is much more involved here. The reviewer called upon the counsel of one of his colleagues, R. E. Hirsch, who concurred that there was no language barrier here, but a difference in substance concerning the art of data processing in the United States and Germany. The paper then contains a description of what is meant by electronic data processing, program control, machine learning and thinking, and cybernetics. The author then discusses the contribution of electronic data processing machines to statistical analyses, such as pre-

editing data, their ability to analyze data in greater detail with increased precision in a given period of time due to the increased computing speed. The author does not seem to be aware that there are other aspects where digital computers can play an important role in statistics, namely, areas which the reviewer refers to as administrative control and simulation. While the acceptance and utilization of digital computers in Germany may be several years behind that of the Americans and English, the paper would indicate an active interest in the use of computers for statistical calculation.

M. Muller (New York)

6094:

Schäfer, Hans-Willy. Der Einsatz von elektronischen Rechenanlagen bei Unternehmen, insbesondere für statistische Arbeiten. Allg. Statist. Arch. 43 (1959), 354-356.

The author discusses use of a digital computer, "without magnetic tapes", in a closed shop computation center. He gives two examples or problems apparently worked out. The first "shows that electronic digital computers save large batteries of punched card equipment". In the second description he describes listing of tables in a memory that is comparable to usual procedures already developed elsewhere, and points out the usefulness of combining preliminary sorting with digital computing machine runs. The author's experience, in an insurance firm, should be of interest to any reader of German collecting case histories of data processing procedures, since the paper is of a generally higher caliber than most found in typical U.S. popular data processing journals, for example.

J. W. Carr, III (Chapel Hill, N.C.)

6095:

Koller, Siegfried. Der Einfluss der Automatisierung auf die Aufgabenstellung der Statistik. Allg. Statist. Arch. 43 (1959), 363-368.

This expository paper discusses the influence of "automation" on the design of statistical projects. It appears that this writer uses the term automation in the sense of Szameitat's paper reviewed above. The author reviews the possibility of having computer applications which, in fact, now exist here in the United States; for example, non-linear estimation techniques, considering problems with a large number of independent variables, and problems of processing large volume data subject to data reduction techniques. The author makes the obvious interesting observation that as the means of doing more rapid and precise data processing improves, the domain of the types of interesting questions that can be considered is correspondingly enlarged.

M. Muller (New York)

6096:

Zindler, Hans-Joachim. Sachliche und organisatorische Probleme der Programmierung. Allg. Statist. Arch. 43 (1959), 369-377.

This paper provides an independent evaluation of present techniques of using digital computers for large-scale statistical purposes. The author, of the Statistisches Bundesamt, Wiesbaden, makes a good case for closed-shop, professional programmer use of magnetic-tape machines in such large-sized tabulating jobs as population censuses. The author stresses the need for use of flow-diagrams, programmers with subject-matter training and

applied statisticians who know about programming. He has high praise for the techniques of the U.S. Bureau of the Census, and many of the techniques he describes as apparently in use in his own organization are very similar to those in the Bureau. At the time of writing, many of the newest U.S. ideas in automatic programming for data processing problems apparently were not available in Germany; there is only a brief discussion (although favorable) of the use of such techniques. This is a good background source for users as well as beginners in data-processing use of digital computers.

*J. W. Carr, III (Chapel Hill, N.C.)*

6097:

**Marimont, Rosalind B.** Applications of graphs and boolean matrices to computer programming. *SIAM Rev.* 2 (1960), 259-268.

From the author's introduction: "Programming for digital computers is a problem requiring great skill and intensive training, and of such complexity that even the most competent programmers make frequent errors. While this difficulty is somewhat lessened by automatic programming techniques for single computers, it is felt that programming for computer networks will be so much more difficult that, in order to be feasible, the logic computer programming must be put on a firm mathematical basis. Graphs and boolean matrices seem to be the best mathematical tools for this purpose." This expository article also demonstrates some graphical and matricial techniques for testing the consistency of a precedence relation.

*F. Harary (Ann Arbor, Mich.)*

6098:

**Saunders, David R.** A computer program to find the best-fitting orthogonal factors for a given hypothesis. *Psychometrika* 25 (1960), 199-205.

The paper has a brief, and frank, discussion of some of the difficulties confronting those interested in the problem of rotation in factor analysis. The author estimates that at least fifteen (not necessarily distinct) approaches to the problem of rotation in factor analysis now exist which utilize digital computers. Without going into detail, a computer program which some of his associates programmed for an IBM 650 is discussed. This program utilizes a modification of the quartimax computation for factor rotation. The computer program permits a hypothesized factor pattern to be included along with the data and the machine program computes a sum of squares and a sum of fourth powers which in "some non-statistical sense" measure the fit of the data to the hypothesized factor pattern.

*M. Muller (New York)*

6099:

**Dimsdale, B.; Weinberg, G. M.** Programmed error correction in Project Mercury. *Comm. ACM* 3 (1960), 649-652.

This paper shows how the effective tape reliability of a computer can be improved by extra coding and extra tape storage in the form of a Hamming error-correcting code. The results obtained may be summarized: "At random times, quite infrequently, an entire word from tape would fail to reach its proper location in memory, without activating the machine's redundancy checking. When this

error occurred in other programs, they were of course completely disabled. Our error correction, however, automatically replaced the entire word and allowed the program to proceed. Ordinarily, we would not even notice such errors, but we were especially vigilant in order to gather statistics for this paper. To date, we have not had a single system failure caused by an uncorrectable tape error."

"Automatic error correction has also shown to advantage in a type of data transmission we use frequently; namely, the physical shipment of tapes. We have shipped dozens of these ECC tapes without ever experiencing that unspeakable frustration caused by uncorrectably bad tapes 800 miles from the nearest duplicates."

These results do not depend on the machine as much as on the coding used to write the error correction system.

*R. W. Hamming (Stanford, Calif.)*

6100:

★**Annual review in automatic programming. Vol. I.** Papers read at the Working Conference on Automatic Programming of Digital Computers held at Brighton, 1-3 April 1959. Edited by Richard Goodman. International Tracts in Computer Science and Technology and Their Application, Vol. 3. Pergamon Press, New York-Oxford-London-Paris, 1960. xi+300 pp. \$10.00.

This is the first volume of an annual series designed to record the most important contributions from all quarters in the field of automatic programming.

The book is dedicated to Alan M. Turing who, as A. D. Booth remarks in his opening address, first enunciated the fundamental theorem upon which all studies of automatic programming are based and whose two papers on computable numbers are reprinted as an appendix. In "Future trends in automatic programming", E. A. Glennie discusses the environment of a computer, the concepts of machine languages and automatic codes, and outlines features, advantages and disadvantages of present automatic programming languages; as their anticipated major improvement he quotes the increase in the use of declarative statements and describes possible future uses of such statements. K. A. Redish, in "Some problems of a universal autocode" presents arguments in favor of standardized notations for programming. In "The Mark 5 system of automatic coding for TREAC", P. M. Woodward describes a complete algebraic programming system for the computer at the Royal Aircraft Establishment. In "Assembly, interpretive and conversion programs for PEGASUS", G. E. Felton describes this computer and a program, called Initial Orders, which is permanently stored in an isolated part of the main store. It is the principal means for the input of programs and incorporates an assembly routine for dealing with the selection of library subroutines and for the building up of a complete program from its component parts. Details of library subroutines and an interpretive routine for handling complex numbers in floating-point form are also given. Operational experience with this system is next discussed, and two case histories are given, by W. F. M. Payne. The same system, primarily intended for scientific applications, is next shown, by P. M. Ronaldson, to be applicable to business data processing, with a sales analysis and forecasting program taken as an example. The next paper is "The application of formula translation to the automatic coding of ordinary differential equations" by J. P. Cleave.

The author states that the investigation undertaken in this paper was motivated by the following observation: "Once a method of solution of a problem has been decided upon, there is enough information embodied in the mathematical formulation of that solution—expressed in the usual mathematical symbolism—to enable a program to be constructed automatically, without the intermediary of a universal code." The application of the principle to ordinary differential equations is discussed in this paper. "MERCURY Autocode—Principles of the program library", by R. A. Brooker, describes the arrangements for using programs within a larger program. "Automatic programming of DEUCE" by C. Robinson and "Further DEUCE interpretive programs and some translating programs" by S. J. M. Denison are concerned with various schemes that have been developed to assist in programming of DEUCE.

R. J. Ord-Smith in "The STANTEC-ZEBRA Simple Code and its interpretation", describes this code, which is an interpreted code designed to combine the advantages of a really simple form of code for the beginner with further facilities giving a comprehensive and flexible programming code. "The Share Operating System for the IBM 709" by K. V. Hanford describes this complete integrated system for programmer communication with the IBM 709. S. Gill, in "The philosophy of programming", discusses in a general way programming, phases in its development and the requirements of an automatic coding scheme. "Automatic programming and business applications" are discussed by G. Cushing and developments in this area in connection with various computers commented upon. "The FLOW-MATIC and MATH-MATIC automatic programming systems", designed for UNIVAC, are discussed by A. Taylor, and "TIDE: a commercial compiler for the IBM 650" is described by E. Humby. J. E. Meggitt, in "Auto-programming for numerically controlled machine tools", gives an account of a programming scheme whereby a Ferranti Mark I prepares punched tapes for control of a milling machine manufacturing accurate models for wind tunnel tests.

It is gratifying that, after a slow start, the art of automatic programming is receiving sufficient attention in Great Britain to justify the publication of these handsome Annual Reviews. It is less cheerful to observe that the very wealth of different coding systems reported on in these pages indicates how far removed we still are from the production of a true "Universal Autocode", which should be the *raison-d'être* of much of this effort.

W. Freiburger (Providence, R.I.)

6101:

★Deuxièmes Journées Internationales de Calcul Analogique, Strasbourg, 1-6 sept. 1958. Actes. Presses Académiques Européennes, Brussels, 1959. xii + 501 pp. 158s.; \$22.00.

This volume contains the papers which were read at the conference mentioned in the title. The various sections are: (1) Description of machines using electronic techniques; (2) Non-linear elements, memory, and various types of one- and two-dimensional function generators; (3) Accuracy and stability of simulation methods; (4) Mechanical and electro-mechanical calculators; (5) Magnetic and resistance analog computers; (6) Analog-digital techniques; (7) Applications.

There is little that is new and the mathematical content

is relatively low but the volume may interest mathematicians who wish to have a concise statement of the type of problem which arises in such things as power networks, nuclear power reactors, process control, and Wigner stress release in reactor control rods. The papers are distributed between the French and English languages in the ratio of about one to one, but the French papers are considerably more accurately produced than the English ones, in which numerous editorial or printing errors tend to annoy the reader. A. D. Booth (London)

6102:

Boscher, Jean. Application de la méthode des effets élémentaires aux réseaux superposés. C. R. Acad. Sci. Paris **250** (1960), 4097-4099.

This paper extends the network analogue computer techniques previously suggested by the author for solving Laplace type (second order) partial differential equations [same C. R. **250** (1960), 448-450; MR **22** #2045] to fourth order partial differential equations of the biharmonic type.

J. G. L. Michel (Teddington)

#### MECHANICS OF PARTICLES AND SYSTEMS

See also A5551, A5934, 6575, 6576.

6103:

Luttinger, J. M.; Thomas, R. B., Jr. Variational method for studying the motion of classical vibrating systems. J. Mathematical Phys. **1** (1960), 121-126.

The action integral for a periodic one-dimensional classical mechanical system is shown to be stationary for small deviations in both the coordinate and period for fixed energy,  $E$ . By choosing a suitable trial function for the coordinate,  $q$ , as a function of  $E$  and  $t$ , this variational principle can be used to obtain "surprisingly accurate" values for the frequency and Fourier coefficients of  $q$ . Several examples are discussed: potential energy  $V = \frac{1}{2}|q|^n$ ,  $n > 0$ , the simple pendulum, and certain anharmonic oscillators.

D. Falkoff (Waltham, Mass.)

6104:

Atanasiu, Mihail. Considérations sur le calcul des pivots. Bul. Inst. Politehn. București **20** (1958), no. 3, 45-58. (Russian. English and German summaries)

6105:

Romiti, Ario. Problemi di dimensionamento, accoppiamento e fabbricazione di ingranaggi conici per uso universale. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **94** (1959/60), 489-520.

Author's summary: "Sono state esaminate le condizioni limiti d'interferenza e l'estensione delle possibilità d'ingranamento delle ruote qui studiate, tra di loro e con altri tipi di ruote con profilo ad evolvente; si sono quindi descritti due possibili procedimenti per la loro fabbricazione."

6106:

Routh, Edward John. ★Dynamics of a system of rigid bodies. Elementary Part. Being Part I of a treatise on the whole subject. 7th ed., revised and enlarged. Dover Publications, Inc., New York, 1960. xvi + 443 pp. \$2.35.

This edition is an unabridged and unaltered republication of the author's last revised edition published in 1905 [Macmillan, London]. This last revised edition is sometimes referred to as the seventh or eighth edition.

6107:

Glauert, M. B. ★Principles of dynamics. Library of Mathematics. Routledge and Kegan Paul, London, 1960. viii + 80 pp. 5s.

This is a brief but carefully written elementary treatment of classical dynamics. Vector algebra is developed in the first chapter, and applied to the dynamics of particles and systems in the remaining two thirds of the book.

P. Franklin (Cambridge, Mass.)

6108:

Chapman, Seville. Misconception concerning the dynamics of the impact ball apparatus. Amer. J. Phys. 28 (1960), 705-711.

The impact ball apparatus considered is a straight row of equal elastic spheres initially touching each other. When several balls are drawn aside and allowed to strike the remaining larger group, it is commonly stated that certain balls remain at rest. The author shows that on the basis of a linear law of force all the motions can be determined. He checks his theory by calculations and experimental data for the case of three or four balls.

P. Franklin (Cambridge, Mass.)

6109:

Sokolov, Yu. D. Recherches sur la théorie des trajectoires singulières d'un système des points matériels libres. Ukrain. Mat. Ž. 11 (1959), 3-15. (Russian. French summary)

This is an expository paper which summarizes the results obtained in earlier papers by the author [cf. *Osobyie traektorii systemy svobodnykh material'nykh toček*, Akad. Nauk Ukrain. SSR, Kiev, 1951; MR 14, 910].

E. Leimanis (Vancouver, B.C.)

6110:

Adachi, Ryuzo. The motion of a pendulum whose supporting axis moves two dimensionally. Kumamoto J. Sci. Ser. A 4, 192-204 (1960).

The small oscillation in a plane of a rigid pendulum which suffers a damping force and whose supporting point moves in the plane, is discussed under the condition that the pendulum is at the position of the equilibrium initially and the displacement of the supporting point occurs suddenly. The positions of the supporting point  $(x(t), y(t))$  are assumed given. The differential equation for the oscillation angle  $\theta(t)$

$$(1) \quad \ddot{\theta} + 2a\dot{\theta} + b^2\theta = -K\ddot{x} + K\ddot{y}$$

with initial conditions  $\theta(0)=0$ ,  $\dot{\theta}(0)=-\dot{x}(0)/l$ , is treated by a method equivalent to the power series expansion of  $\theta(t)$  with respect to  $\mu$  if we put  $\mu K\ddot{y}$  instead of  $K\ddot{y}$  in (1). A condition for the boundedness of  $\theta(t)$  for  $t \rightarrow +\infty$  is obtained.

T. Kasuga (Osaka)

6111:

Chako, Nicholas. Sur la stabilité des systèmes dynamiques et l'équation de Schrödinger. C. R. Acad. Sci. Paris 251 (1960), 645-647.

The author finds sufficient conditions that a wave equation may be associated with the energy integral of a dynamical system. In particular, it is assumed that the system is (1)  $\dot{x}_i = H_{y_i}$ ,  $\dot{y}_i = -H_{x_i}$ , that  $t$  does not appear explicitly in  $H$ , and that (2)  $H(x_i, y_i) = \frac{1}{2} \sum b^{ik}(x_i) V_{x_i} V_{y_i} + U(x_i) = E$ , where  $V_{x_i} = \dot{x}_i$  and  $E$  is the energy constant. It is further assumed that (1) is stable and that the characteristic exponents associated with certain variational equations based on a periodic solution of (1) all vanish [see Četaev, Prikl. Mat. Meh. 22 (1958), 487-489; MR 21 #4577]. By appropriate changes of variables (2) reduces to the Schrödinger equation if the Einstein relation  $E = h\nu$  is also assumed. The methods and results are closely connected with those of Četaev [op. cit.].

C. S. Coleman (Claremont, Calif.)

6112:

Chako, Nicholas. Sur le passage de l'Optique géométrique à l'Optique ondulatoire. C. R. Acad. Sci. Paris 251 (1960), 852-853.

The author begins with the canonical equations of geometric optics in an isotropic medium: (1)  $dx_i/d\tau = H_{y_i}$ , and  $dy_i/d\tau = -H_{x_i}$  ( $i=1, \dots, n$ ). The methods used in the paper reviewed above are applied to associate with (1) either the wave equation of particle optics,  $\nabla^2 u - \nu^{-2} u_{tt} = 0$ , or else the equation

$$\nabla^2 u + (2\pi i \nu \omega c^{-1} + 4\pi^2 n^2 \nu^2 c^{-2}) u = 0,$$

which resembles an equation given by Louis de Broglie [J. Phys. Radium (6) 7 (1926), 321-337] for wave propagation in a dispersive medium.

C. S. Coleman (Claremont, Calif.)

6113:

Bogusz, Wladyslaw; Szpunar, Kazimierz. System of one degree of freedom with strong non-linear even damping. Rozprawy Inż. 8 (1960), 187-200. (Polish. Russian and English summaries)

The qualitative investigation of the behavior of a system governed by the equation  $\ddot{x} + \omega^2 x + \alpha \dot{x}^3 = 0$ ,  $\alpha > 0$ , is made by means of velocity hodograph, and the quantitative analysis is made using the W. Bogusz method [Arch. Mech. Stos. 11 (1959), 691-713; MR 22 #3856], in the  $\dot{x}, x$  plane. It is found that the solutions are periodic in the neighborhood of singular points, and infinite outside of this region. The form of the trajectory is determined in the whole phase plane. The expressions for the periods and the amplitudes are also calculated.

R. M. Evan-Iwanowski (Syracuse, N.Y.)

# STATISTICAL THERMODYNAMICS AND MECHANICS

See also 6457.

6114:

Peterlin, Anton. Deviations from equipartition of energy with nonquadratic Hamiltonian. Amer. J. Phys. 28 (1960), 716-718.

Author's summary: "If the Hamiltonian contains non-quadratic terms, the energy contribution per particle and

degree of freedom deviates from  $\frac{1}{2}kT$  (classical approximation). A very simple case which can be easily treated is that of a particle in a sinusoidal potential field. Energy, free energy, and molar heat are calculated as an example of the resulting deviations from equipartition."

6115:

Alder, B. J.; Wainwright, T. E. Studies in molecular dynamics. I. General method. *J. Chem. Phys.* **31** (1959), 459-466.

Authors' summary: "A method is outlined by which it is possible to calculate exactly the behavior of several hundred interacting classical particles. The study of this many-body problem is carried out by an electronic computer which solves numerically the simultaneous equations of motion. The limitations of this numerical scheme are enumerated and the important steps in making the program efficient on the computers are indicated. The applicability of this method to the solution of many problems in both equilibrium and nonequilibrium statistical mechanics is discussed."

6116:

Alder, B. J.; Wainwright, T. E. Studies in molecular dynamics. II. Behavior of a small number of elastic spheres. *J. Chem. Phys.* **33** (1960), 1439-1451.

Authors' summary: "The equation of state and the collision rate for systems ranging in size from four to 500 particles are described. The dependence of the results on the number of particles is qualitatively discussed and in this way insight is gained as to what is required of more accurate analytical theories. By comparing the results to various analytical theories now available their region of validity is established. The number of particles necessary at various densities to obtain a quantitative description of the equilibrium properties is delineated. Whether a first-order phase transition exists for hard spheres remains open until larger systems are investigated."

6117:

Huang, Kerson. Equation of state of a Bose-Einstein system of particles with attractive interactions. *Phys. Rev.* (2) **119** (1960), 1129-1142.

From the author's summary: "The equation of state in the grand canonical ensemble is calculated for a system of Bose-Einstein particles with hard-sphere repulsive interactions and weak long-range attractions. The energy levels used in this calculation are modified forms of those derived in an earlier paper. The calculation is carried out in the limit of no interactions, and attention is focused on the thermodynamic phases of the system. It is shown that the gross features of the equation of state of  $\text{He}^4$  are reproduced. There are the phases: gas, liquid I, and liquid II. The phase transition between gas and the two liquids are first order transitions. The transition terminates in a critical point. The transition between liquid I and liquid II is the Bose-Einstein condensation. Liquid II has a negative coefficient of thermal expansion. Across the transition between liquids I and II the specific heat is discontinuous in value. In the limit of no interactions, the critical point recedes towards zero temperature, zero pressure, and infinite volume."

R. M. Evan-Iwanowski (Syracuse, N.Y.)

6118:

Bowers, D. L.; Salpeter, E. E. Correction to the Debye-Hückel theory. *Phys. Rev.* (2) **119** (1960), 1180-1186.

Authors' summary: "The problem of a gas of particles all of the same charge, imbedded in a neutralizing medium of uniformly distributed charge of the opposite sign, is considered in terms of classical statistical mechanics. If a dimensionless parameter  $\epsilon$ , roughly the inverse of the number of particles contained inside a Debye sphere, is small compared to unity, the Debye-Hückel theory is a good first approximation. For this case the corrections in the next order in  $\epsilon$  are derived for the potential of mean force and the interaction energy. It is shown how this correction has to be modified for very small particle separation; the expansion in powers of  $\epsilon$  is not strictly a Taylor expansion and factors such as  $\ln \epsilon$  appear in the higher terms. Methods are given for numerical calculation of some auxiliary functions even when the parameter  $\epsilon$  is not small."

6119:

Parker, J. G. Rotational and vibrational relaxation in diatomic gases. *Phys. Fluids* **2** (1959), 449-462.

Author's summary: "Calculations of rotational and vibrational relaxation times for gases composed of homonuclear diatomic molecules have been carried out. The model used for the molecular interaction potential consists of an attractive component, which acts between geometrical centers of the molecules, and a repulsive component which is assumed to originate from two centers of force in each of the molecules. For large intermolecular separations, the attractive forces prevail while at close distances the repulsive forces control. Using this model, the number of collisions to establish rotational equilibrium  $Z_R$  and also the number for vibration  $Z_V$  are calculated. Both  $Z_R$  and  $Z_V$  contain some of the same molecular parameters and are therefore dependent on each other. From the analysis it turns out that  $Z_R$  is a gradually increasing function of increasing temperature and  $Z_V$  is a rapidly decreasing function of increasing temperature. Comparison with experiment for the gases chlorine, nitrogen, and oxygen indicates that the calculations are for the most part reliable."

H. L. Frisch (Murray Hill, N.J.)

6120:

Vallander, S. V. New kinetic equations in the theory of monatomic gases. *Dokl. Akad. Nauk SSSR* **131** (1960), 58-60 (Russian); translated as *Soviet Physics. Dokl.* **5**, 269-271.

This paper deals with the kinetic equation to determine the molecular distribution function  $f(r, u, t)$  of a gas flowing around a moving body. This can be expressed in terms of the probability of collision in the gas and that on the surface of the body and the probability that a free motion of a gas molecule persists for a given time interval. This idea leads to a set of integral equations for these probabilities which contains the boundary conditions on the surface.

R. Kubo (Tokyo)

6121:

Vineyard, George H. Neutron scattering by fluids and the law of corresponding states. *Phys. Rev.* (2) **119** (1960), 1150-1153.

It is shown that the law of corresponding states implies correspondences in the pair-correlations in fluids. One can thus derive the scattering cross sections for neutron and X-ray scattering of all fluids of the class to which the law of corresponding states holds from the scattering data for one fluid from this class.

*D. ter Haar (Oxford)*

6122:

Morita, Tohru. Theory of classical fluids: hyper-netted chain approximation. III. A new integral equation for the pair distribution function. *Progr. Theoret. Phys.* **23** (1960), 829-845.

For part II see *Progr. Theoret. Phys.* **21** (1959), 361-382 [MR **21** #4590].

This paper introduces an improvement to the "hyper-netted chain" approximation, discussed in the first paper [ibid. **20** (1958), 920-938; MR **21** #1722] of the series. Expressions for the chemical potential free energy and equation of state are obtained with the help of the new approximation. To compare with other approximations, and in particular that implicit in the Born-Green linear integral equation, the virial coefficients are calculated. It is found that the improved hypernetted chain approximation is superior in taking exact account of a greater variety of bond-figures.

*H. S. Green (Adelaide)*

6123:

Richardson, J. M.; Brinkley, S. R., Jr. Cell method in grand canonical ensemble. *J. Chem. Phys.* **33** (1960), 1467-1478.

This is a treatment of the cell model of a liquid, based on the use of the grand partition function instead of the canonical partition function, which is more usual. The authors apply a variational procedure to determine the best values of the thermodynamic function consistent with (a) the neglect of correlations between neighbouring cells and (b) the neglect of multiple occupancy. In this way they recover results obtained by earlier workers in this field. Towards the end of the paper they develop what is, in effect, a cell cluster theory which takes account of correlations, but do not relate their results to those which have been obtained by other methods.

*H. S. Green (Adelaide)*

6124:

Helfand, E.; Reiss, H.; Frisch, H. L. Scaled particle theory of fluids. *J. Chem. Phys.* **33** (1960), 1379-1385.

In this paper the authors derive a number of exact mathematical results for the statistical mechanics of fluids. The method introduces the coupling of one molecule (called a  $\lambda$ -cule) to the others by a fictitious potential of the form  $u(r/\lambda)$ , where  $u(r)$  is the actual potential energy of two molecules. Many properties of the fluid are found to depend on a function  $\Theta$  of the density, temperature, and the parameter  $\lambda$ . This function is incompletely determined by several conditions which are derived in the course of the paper. Attention is directed to special results for rigid spherical molecules, and for intermolecular forces with a limited range.

*H. S. Green (Adelaide)*

6125:

Meeron, Emmanuel. Nodal expansions. III. Exact integral equations for particle correlation functions. *J. Mathematical Phys.* **1** (1960), 192-201.

[For part II see Meeron and Rodemich, *Phys. Fluids* **1** (1958), 246-250; MR **22** #3162.]

Author's summary: "The density expansions of the pair distribution function and potential of average force are analyzed topologically in terms of cutting points and bifocal points. The analysis leads to conversion of the expansions into series with cluster integrals involving products of the total correlation functions,  $h(r) = g(r) - 1$ , at finite density, rather than the usual zero-density Ursell  $f$ -functions. An integral equation for the pair potential of average force and the pair distribution function is thus obtained. The equation is formally exact and closed in pair space, involving no triplet distributions such as occur in the treatments of Kirkwood and Yvon-Born-Green. Solution of the equation also yields directly the Ornstein-Zernike direct correlation function. Equations for the free energy in terms of the direct correlation function are presented, thus providing a unified and self-consistent treatment of all thermodynamic properties of a many-body system. The relation of the new equation to the Ornstein-Zernike theory of liquids and to phase transitions is discussed. The possibility of derivation for condensed phases is briefly noted. A simple approximation, involving only the convolutive terms in the cluster expansions of correlation functions, is proposed."

*H. Mori (Kyoto)*

6126:

Fisher, M. E. Lattice statistics in a magnetic field. I. A two-dimensional super-exchange antiferromagnet. *Proc. Roy. Soc. London. Ser. A* **254** (1960), 66-85.

Author's summary: "The partition function of a two-dimensional 'super-exchange' antiferromagnet in an arbitrary magnetic field is derived rigorously. The model is a decorated square lattice in which magnetic Ising spins on the bonds are coupled together via non-magnetic Ising spins on the vertices. By use of the decoration transformation all the thermodynamic and magnetic properties of the model are derived from Onsager's solution for the standard square lattice in zero field. The transition temperature  $T_t(H)$  is a single-valued, decreasing function of the field  $H$ . The energy and the magnetization are continuous functions of  $T$  for all magnetic fields; but the specific heat and the temperature gradient of the magnetization become infinite as  $-\ln|T - T_t|$ . The initial ( $H = 0$ ) susceptibility is a continuous and smoothly varying function of  $T$  with a maximum 40% above the critical point; but  $\partial\chi/\partial T$  becomes infinite at  $T = T_c$ . In a non-vanishing field the susceptibility has a logarithmic infinity at  $T = T_t$ . For small fields the behaviour near the critical point is given by

$$\chi \approx (N\mu^2/kT) \times \{2 - 2^{1/2} - D(T - T_c) \ln|T - T_c| - D'H^2 \ln|T - T_c|\},$$

where  $D$  and  $D'$  are constants."

*S. Sherman (Detroit, Mich.)*

6127:

Fisher, M. E. Lattice statistics in a magnetic field. II. Order and correlations of a two-dimensional super-exchange antiferromagnet. *Proc. Roy. Soc. London. Ser. A* **256** (1960), 502-513.

Author's summary: "The long-range order and pair correlation functions of a two-dimensional superexchange antiferromagnet in an arbitrary magnetic field are derived

rigorously from properties of the standard square Ising lattice in zero field. (The model investigated was described in part I: it is a decorated square lattice with magnetic spins on the bonds coupled antiferromagnetically via non-magnetic spins on the vertices.) The behaviour near the transition temperature in a finite field is similar to that of the normal plane lattice, i.e., the long-range orders or spontaneous magnetizations of the sublattices vanish as  $(T_c - T)^{1/8}$  and the pair correlations behave as

$$\omega_c + W(T - T_c) \ln |T - T_c|.$$

The configurational entropy is discussed and the anomalous entropy in the critical field at zero temperature is calculated exactly." *S. Sherman* (Detroit, Mich.)

6128:

**Pressman, Walter; Keller, Joseph B.** Equation of state and phase transition of the spherical lattice gas. *Phys. Rev.* (2) **120** (1960), 22-32.

The authors introduce a "spherical lattice gas" whose relation to the Berlin-Kac spherical model of a ferromagnet is analogous to the relation shown by Yang and Lee between a lattice gas and the Ising model of a ferromagnet. The "spherical lattice gas" like the Berlin-Kac model lends itself to explicit calculation even when the dimension is three and also exhibits unphysical behavior in some regions. Phase transition curves and isothermals are presented. *S. Sherman* (Detroit, Mich.)

6129:

**McKay, M. H.; Keane, A.** A correction to the effective resonance integral in heterogeneous nuclear reactors to allow for fuel geometry. *Austral. J. Appl. Sci.* **11** (1960), 1-15.

An improvement in the calculation of the "surface absorption" term in the Wigner formulation of the problem of the title. The improvement stems from the use of an improved approximation to the experimental and the error function in the evaluation of certain integrals which arise in this problem. *R. R. Coveyou* (Oak Ridge, Tenn.)

6130:

**Cook, J. L.; Elliott, D.** The tabulation of three functions arising in nuclear resonance theory. *Austral. J. Appl. Sci.* **11** (1960), 16-32.

This paper contains tables of certain functions useful in the calculation of the effect of moderator temperature (Doppler effect) on neutron cross sections in a material whose cross section is described by a single-level Breit Wigner resonance cross section formula. The tables are more extensive than those previously published. *R. R. Coveyou* (Oak Ridge, Tenn.)

6131:

**Davison, B.** On the rate of convergence of the spherical harmonics method. (For the plane case, isotropic scattering.) *Canad. J. Phys.* **38** (1960), 1526-1545. (English, German and Russian summaries)

This paper is a careful and skilful examination of the relationship between the solution of the Boltzmann equation of neutron transport and the solutions of the spherical harmonic approximations to the same equation.

As an example of the type of conclusion reached we quote a result; that the error in the critical thickness of a "one velocity slab reactor", calculated in the " $P_N$  approximation" is:

$$a - a_N = \frac{-c\pi^2}{48(N+3/2)^2} [1 + O(\log[2N+3]/N)],$$

where  $a$  is the exact,  $a_N$  the approximate critical half-thickness. The procedure used is to show that the solution of the " $P_N$  approximation" satisfies an integral equation with a kernel only slightly different from the kernel of the integral equation satisfied by the exact solution, and to analyze the difference in solutions, eigenvalues, etc., imposed by this difference in the kernels.

*R. R. Coveyou* (Oak Ridge, Tenn.)

6132:

**Sarma, G.** Diffusion des neutrons lents par l'hydrogène liquide. *J. Phys. Radium* **21** (1960), 783-788. (English summary)

Author's summary: "The differential slow neutron scattering cross section by liquid hydrogen is expressed under some reasonable assumptions, as a product of two factors of simple physical meaning. The first factor, throughout spin correlations, involves transitions between molecular states and finally leads to a kind of an angular dependent scattering length, which is computed exactly for every transition in the rigid rotator approximation. One is then left with a second factor depending only on the translational motion of the molecules, which turns out to be a Fourier transform of the well-known pair correlation function for a monatomic liquid.

"It is shown that what one might call the 'coherent' part of the scattering, i.e., the part involving correlation between pairs of distinct molecules, is negligible.

"So the scattering is mainly related to the so-called self correlation function. Recoil appears naturally as a prominent feature. It is shown that the scattering is generally well described by means of a perfect gas model, taking into account the recoil of the molecules. This model leads to well-separated peaks at a given angle. The small angle elastic scattering, where the perfect gas model is no longer valid, should allow for the measurement of a coefficient of self diffusion."

6133:

**Katz, Stanley.** Best temperature profiles in plug-flow reactors: methods of the calculus of variations. *Ann. New York Acad. Sci.* **84**, 441-478 (1960).

6134:

**Golden, Sidney.** Statistical theory of electronic energies. *Rev. Mod. Phys.* **32** (1960), 322-327.

The author is concerned with obtaining energies of many electron atoms by taking averages over density matrices. He shows that the advantages of working with the density matrix rather than the Schrödinger equation are twofold: (i) the former can be expressed in a form invariant to choice of basis, and (ii) a systematic approximation to the density matrix need only entail a suitable expansion of the exponential of the Hamiltonian. The Fermi-Thomas result is gotten from the lowest order approximation. Subsequent approximations are indicated and numerical results cited. *D. Falkoff* (Waltham, Mass.)

6135:

Hirschfelder, Joseph O. Classical and quantum mechanical hypervirial theorems. *J. Chem. Phys.* **33** (1960), 1462-1466.

Let  $W$  be a dynamical variable and  $H$  the Hamiltonian for  $N$  particles. Then, in classical mechanics the time average of the Poisson brackets  $\langle (H, W) \rangle = 0$ , while in a quantized system the expectation value of  $J(W) = (i/\hbar) \cdot [WH - HW]$  vanishes for any state  $\psi$  such that  $H\psi = E\psi$ . Two choices of  $W$  are discussed in detail:  $W = f(q_1, \dots, q_{3N})$  and the more interesting case  $W = f(q_1, \dots, q_{3N})p_k$ . Since the latter form of  $W$  generalizes the virial theorem of Clausius (for which  $f = q_k$ ), it is said to lead to hypervirial theorems. The author shows that for any real  $\psi$  that satisfies  $\int \psi J(f p_k) \psi d\tau = 0$  for all  $f$ , and which decays sufficiently fast, it follows that  $H\psi = E\psi$ . This result may be used to select approximate wave functions by choosing  $f$ 's that are sensitive to properties of interest.

J. R. Klauder (Murray Hill, N.J.)

6136:

Sher, A.; Primakoff, H. Approach to equilibrium in quantal systems: magnetic resonance. *Phys. Rev.* (2) **119** (1960), 178-207.

This paper is devoted to the problem of deriving the master equation, that is, the transport equation for the probabilities, from the Schrödinger equation, that is, the equation of motion of the probability amplitudes. The master equation is first derived for the system of interest plus a heat bath, then for the system of interest by itself, and finally for an individual particle in the system of interest. The various approximations used in the derivations are carefully discussed. The different master equations are then solved and the spectrum of relaxation times is discussed and a comparison is made with the "spin-temperature" procedure. Finally, the case of magnetic resonance is analyzed: the time variation of the magnetization, and the transition probabilities involved are discussed, as well as the inapplicability of a master equation to a rigid lattice and the importance of quantal coherence effects which are neglected in the derivation of the master equation.

D. ter Haar (Oxford)

6137:

Fradkin, E. S. Some general relations in statistical quantum electrodynamics. *Ž. Eksper. Teoret. Fiz.* **38** (1960), 157-160 (Russian. English summary); translated as *Soviet Physics. JETP* **11**, 114-116.

In quantum electrodynamics a number of relations known as Ward's identity and its generalizations follow from the gauge invariance of the theory. In this paper, the author investigates the validity of the analogous relations in "statistical quantum electrodynamics" and shows that, with suitable adaptations, those relations are still valid. Statistical quantum electrodynamics differs from ordinary quantum electrodynamics in that the Green's functions are defined as grand canonical averages instead of vacuum expectation values.

B. Zumino (New York)

6138:

Saitō, Nobuhiko. Theory of irreversibility. *Phys. Rev.* (2) **117** (1960), 1163-1173.

The asymptotic behaviour is discussed for the after-effect function of an isolated system which represents the response of the system to a single pulse of an electric field. The transfer function,  $\Phi(z)$ , defined as the Laplace transform of the after-effect function, will have poles on the left-half plane of  $z$  in the limit of a large system, which implies an irreversible decay of the after-effect. This is shown by applying the analogy of two-dimensional electrostatic problems, but would require more mathematical analysis. The concept of entropy production is also discussed.

R. Kubo (Tokyo)

6139:

Magalinskii, V. B.; Terleckii, Ya. P. An equation for the diffusion in phase space for nonlinear systems. *Ž. Eksper. Teoret. Fiz.* **36** (1959), 1731-1735 (Russian); translated as *Soviet Physics. JETP* **9**, 1234-1236.

A method developed by the authors in earlier papers [same *Ž.* **34** (1958), 729-734; *MR* **20** #4942, see also #6140 below] is used to derive an equation of motion for the probability density in phase space for arbitrary non-linear systems. The equation obtained is the usual Fokker-Planck equation if one takes the limiting case of a linear system.

D. ter Haar (Oxford)

6140:

Magalinskii, V. B. Dynamical model in the theory of the Brownian motion. *Ž. Eksper. Teoret. Fiz.* **36** (1959), 1942-1944 (Russian. English summary); translated as *Soviet Physics. JETP* **9**, 1381-1382.

A system consisting of a one-dimensional harmonic oscillator which is coupled to a set of one-dimensional harmonic oscillators is considered, especially with respect to the random forces acting upon the oscillator. The Nyquist formula which relates the correlation of the fluctuating force with the dissipative properties of the system is derived for this particular case.

D. ter Haar (Oxford)

6141:

Uhlhorn, U. Onsager's reciprocal relations for non linear systems. *Ark. Fys.* **17**, 361-368 (1960).

For systems subject to large deviations from thermodynamic equilibrium, the Onsager phenomenological coefficients  $L$  depend upon the thermodynamic coordinates  $\alpha$  which measure the extent of the deviation. The author shows that in this situation the phenomenological relations for the time derivatives of the  $\alpha$ 's must be modified to read

$$(1) \quad \frac{d\alpha^\mu}{dt} = \frac{1}{f(\alpha)} \frac{\partial}{\partial \alpha^\nu} [L^{\mu\nu}(\alpha) f(\alpha)],$$

where  $f(\alpha)$  is the equilibrium distribution function. The coefficients defined by equation (1) obey the Onsager reciprocal relations in the form

$$(2) \quad L^{\mu\nu}(\alpha) = \pm L^{\nu\mu}(T\alpha),$$

the plus sign prevailing if the product  $\alpha^\mu \alpha^\nu$  is even with respect to time reversal, the minus sign if it is odd; the symbol  $T$  appearing in the argument on the right-hand side of equation (2) is the time reversal operator.

S. Prager (Minneapolis, Minn.)

## ELASTICITY, PLASTICITY

See also A5784, A5834, A5835, 6088, 6594.

6142:

Adkins, J. E. Further symmetry relations for transversely isotropic materials. *Arch. Rational Mech. Anal.* **5**, 263-274 (1960).

The author solves the problem of determining the form of functional dependence of an asymmetric second order tensor on any number of similar tensors and vectors, assuming the functional form is invariant under rotations about a direction. The underlying space is assumed to be three-dimensional. *J. L. Ericksen (Baltimore, Md.)*

6143:

Seth, B. R. Finite bending of plates into cylindrical shells. *Ann. Mat. Pura Appl.* (4) **50** (1960), 119-125.

Using a linear stress-strain relation the surface tractions required to bend a plate into a cylindrical shell, whose section is given by part of the curve  $r^n \cos n\theta = a^n$ , are computed, where  $n$  and  $a$  are constants and  $r, \theta$  are plane polar coordinates. *A. E. Green (Newcastle-upon-Tyne)*

6144:

Grioli, Giuseppe. Elasticità asimmetrica. *Ann. Mat. Pura Appl.* (4) **50** (1960), 389-417.

This paper differs from the usual treatments of finite deformation elasticity in that the presence of body couples and of boundary surface couples is postulated, and in that the forces on a small element of area in the interior are assumed to be statically equivalent to a couple in addition to a force. Additional equations of equilibrium are required and the expressions for the thermodynamic functions contain extra terms. For small displacements the analogues of the generalised Hooke's law are derived and a hypothetical example is given.

*D. R. Bland (Manchester)*

6145:

Babuška, Ivo; Rektorys, Karel; Vyčichlo, František. *★Mathematische Elastizitätstheorie der ebenen Probleme.* In deutscher Sprache herausgegeben von Winfried Heinrich. Akademie-Verlag, Berlin, 1960. xii + 478 pp. DM 96.00.

German translation of the Czech original [Naklad. Českoslov. Akad. Věd, Prague, 1955] reviewed in MR 17, 1253 and MR 19, 480.

6146:

Isida, Makoto. On some plane problems of an infinite plate containing an infinite row of circular holes. *Bull. JSME* **3** (1960), 259-265.

The author deals with plane problems of a plate with a row of circular holes, subject to uniform tension, tension in arbitrary direction, uniform shear, bending in plane, uniform internal pressure, cosine- and sine-types of internal pressure. Reference is made to the well-known work by Howland, but the author seems not familiar with the more recent similar work [e.g., K. J. Schulz, Thesis, Delft, 1941; Nederl. Akad. Wetensch. Proc. **45** (1942), 233-239, 341-346, 457-464, 524-532; **48** (1945), 282-291, 292-300; MR 5, 250; 7, 503].

*W. T. Koiter (Delft)*

6147:

Buecker, Hans F. Some stress singularities and their computation by means of integral equations. Boundary problems in differential equations, pp. 215-230. Univ. of Wisconsin Press, Madison, 1960.

The author investigates the effect of a crack perpendicular to the boundary of an infinite strip. The problem is formulated in terms of an integral equation which is solved approximately. The coefficient of stress singularity at the root of the crack is obtained with adequate accuracy.

*W. T. Koiter (Delft)*

6148:

Atsumi, Akira. Stresses in a circular cylinder having an infinite row of spherical cavities under tension. *J. Appl. Mech.* **27** (1960), 87-92.

The author gives the solution of the title problem by first constructing a sufficiently general periodic stress function satisfying the boundary conditions on the surface of the cylinder and next obtaining the unknown coefficients from the boundary condition on one spherical cavity. The resulting infinite set of equations is solved approximately by a perturbation method in which  $\lambda$  (the ratio of hole radius to cylinder radius) is the small parameter. Convergence of the perturbation method appears to improve with increasing distance between cavities. Numerical results are compared with Ling's results for a single cavity [Quart. Appl. Math. **13** (1956), 381-391; MR 19, 902] and with analogous problem of circular holes in a strip.

*W. T. Koiter (Delft)*

6149:

Nomura, Yasuo. Plane stresses of the belt which has a discontinuity. (The case having a rectangular projection with sharp corners and rounded off corners on one side). *Bull. JSME* **3** (1960), 247-254.

This paper deals with the plane stress of an infinite isotropic strip having a rectangular notch in one edge with either sharp or rounded off corners where it meets the strip. A Schwarz-Christoffel transformation  $z = z(\zeta)$  of the notched strip on to the unit circle  $|\zeta| = 1$  is used and expressed in the form of an infinite power series. Stress functions are determined in terms of this series when the strip is under tension at infinity. The mapping function  $z(\zeta)$  has singular points at the sharp corners and the edge stress becomes infinite at these points, but for the rounded off corners there are no singular points and, although the convergence of the series is not very rapid, it is possible to calculate the edge stresses. Comparison is made with experimental results using photo-elasticity and the wire-strain meter.

*R. M. Morris (Cardiff)*

6150:

Gaziz, D. C. Graphical investigation of geometric aspects of the Hertz problem. *J. Appl. Mech.* **27** (1960), 735-737.

6151:

Leonov, M. Ya.; Čumak, K. I. Pressure under an approximately circular die. *Akad. Nauk Ukrain. RSR. Prikl. Meh.* **5** (1959), 191-199. (Ukrainian. Russian and English summaries)

Authors' summary: "A method is proposed for determining the pressure under the base of a rigid die on an

elastic half-space on condition that the die is approximately circular in shape and that there are no frictional forces between the die and the half-plane.

"This problem is solved by inscribing a circle in the area of contact. The pressure inside this circumference is considered as the pressure under the base of a circular die. The pressure in the remaining part of the contact area is determined from the condition of continuity of pressure under the die at the points of this circumference. As a result, an equation of the Prandtl type is obtained which can be solved by an approximate method. A formula is given for the pressure under the die.

"The problem of determining the pressure of an elliptical die with a flat base on an elastic half-space is considered as an example."

6152:

Collins, Derek. On the stress distributions due to force nuclei in an elastic solid bounded internally by a spherical hollow and in an elastic sphere. *Z. Angew. Math. Phys.* 11 (1960), 3-16. (French summary)

Expressions in infinite series of spherical harmonics are given for the stress due to a force nucleus in a solid with a spherical hollow, and for the stress in a sphere due to two equal and opposite nuclei at internal points; in the first case expressions in closed form are given for the stresses on the boundary. W. R. Dean (London)

6153a:

Kvitka, O. L. Investigation of the stressed and strained state of short thick-walled cylinders subjected to axisymmetrical loading with the aid of computers in the case of arbitrary radial loading. *Dopovidi Akad. Nauk Ukrain. RSR* 1959, 1071-1076. (Ukrainian. Russian and English summaries)

6153b:

Kvitka, O. L. Investigation of the stressed and strained state of short thick-walled cylinders subjected to axisymmetrical loading with the help of computers in the case of arbitrary axial loading. *Dopovidi Akad. Nauk Ukrain. RSR* 1959, 1300-1305. (Ukrainian. Russian and English summaries)

Using the net method in both papers, the author computes tables, which facilitate calculations of the stresses and strains in short thick-walled cylinders subjected to arbitrary axisymmetrical loads. *Z. Kączkowski* (Warsaw)

6154:

Erler, W. Das elastische Verhalten kurzer zylindrischer Gummiprüben. *Hochfrequenztech. Elektroak.* 69 (1960), 170-179.

6155:

Gregory, M. A non-linear bending effect when certain unsymmetrical sections are subjected to a pure torque. *Austral. J. Appl. Sci.* 11 (1960), 33-48.

6156:

Frisch-Fay, R. The deformation of elastic circular rings. *Austral. J. Appl. Sci.* 11 (1960), 329-340.

Author's summary: "A straight, flexible strip is bent into the form of a circular ring and subjected to two equal and opposite radial forces. The discussion is restricted to deformations in the initial plane of the ring. The analysis shows that the elastic line of the deformed ring is always a segment of the nodal or of the undulating elastica. Coordinates corresponding to different values of the load are given and the method is explained by a numerical example."

6157:

Frisch-Fay, R. The flexible leaf spring. *Austral. J. Appl. Sci.* 11 (1960), 341-352.

Author's summary: "Two circular leaf springs facing each other are connected by hinges at their ends and are acted upon by two equal and opposite forces placed symmetrically. The discussion shows that the elastic line of the deformed spring can be replaced by segments of the nodal or of the undulating elastica. Formulae for the coordinates of the elastic line under different loads are given and the method is explained by a numerical example."

6158:

Bassali, W. A. The classical torsion problem for sections with curvilinear boundaries. *J. Mech. Phys. Solids* 8 (1960), 87-99.

Author's summary: "This paper is concerned with the application of complex variable analysis to obtain solutions for the Saint-Venant torsion problem corresponding to certain sections with regular curvilinear boundaries. The area of the cross section is conformally mapped on the unit circle by the transformation  $z = c\zeta / (1 + m\zeta^2 + p\zeta^{2n})$ , involving the parameters  $m, n, p$ . By varying these parameters various shapes having several axes of symmetry are obtained. Sections bounded by  $n (> 2)$  equal and very approximately circular arcs are included as particular cases. Closed and exact expressions are derived for the complex torsion function, the torsional rigidity, and the shearing stresses at any point of the section. The distribution of shearing stress on the boundary is investigated in the general case and graphs showing its variation in four particular examples are plotted."

R. M. Morris (Cardiff)

6159:

Hicks, Raymond. Asymmetrically ring reinforced circular hole in a uniformly end-loaded flat plate with reference to pressure vessel design. *Proc. Inst. Mech. Engrs.* 173 (1959), 329-342.

The author considers an infinite thin plate of thickness  $h$ , with a hole of radius  $b$ . The plate is under a uniform radial stress  $p$  at infinity, and the hole is reinforced by a compact ring having a rectangular cross-section of radial width  $t$ , which is small compared with  $b$ , and of depth  $d > h$ . Let  $w$  be the deflection of the plate at a point of radial coordinate  $r$  and  $\bar{r}$  be the radial stress component acting at the middle plane of the plate, and write:  $\alpha = ph/D$ ,  $\beta = -h\{p - (\bar{r})_b\}b^2/D$ ,  $x = r/\alpha$ ,  $\psi(x) = dw/dr$ ,  $n^2 = 1 + \beta$ , where  $(\bar{r})_b$  is the value of  $\bar{r}$  at  $r=b$  and  $D$  is

the flexural rigidity of the plate. The author first shows that  $\psi(x) = C_2 K_n(x)$ , where  $K_n(x)$  is a modified Bessel function of the second kind and of order  $n$ , and  $C_2$  is a certain constant depending on the values of  $K_n(x)$  and its derivative when  $r=b$ . Since the stresses in the plate are all deductible from  $dw/dr$ , and consequently from  $\psi$ , the author then obtains these stresses in terms of  $K_n(x)$ . The complete solution of the problem then will be given by evaluating the order  $n$  of  $K_n(x)$ . The author obtains this value through a graphical solution of an implicit relation involving  $n$ . Tables are provided by the author to give the numerical values of  $n$  for particular dimensions of the plate.

M. Nassif (Assiut)

6160:

Shiroya, S. On the transverse flexure of a semi-infinite plate with an elliptic notch. *Ing.-Arch.* **29** (1960), 93-99.

Using the Poisson-Kirchhoff plate theory and Muskhelishvili's complex variable method, the author solves the title problem assuming the bending moment to be constant. The effects of the notch are investigated by means of considerable numerical work.

H. D. Conway (Ithaca, New York)

6161:

Holmes, M. Stiffened plating under transverse load. *Quart. J. Mech. Appl. Math.* **12** (1959), 443-453.

Author's summary: "The behaviour, under transverse load, of a flat plate stiffened by beams is analysed by Fourier series and solutions are obtained for symmetrical and antisymmetrical forms of loading.

"An alternative method of analysis is derived for symmetrical loading forms, involving the evaluation of the plate efficiency. This alternative method has some advantage in that the applied loading is expressed in the form of a Fourier series for bending moment rather than load intensity.

"Finally, it is shown that approximate solutions (assuming infinite torsional rigidity of the beams) may be obtained for a wide variety of loading forms by superimposing the solutions for the symmetrical and anti-symmetrical forms of loading."

6162:

Kurata, M.; Okamura, H. Bending of a rectangular plate with two opposite free edges and other two simply supported edges having any clamped portion. *Z. Angew. Math. Mech.* **40** (1960), 310-327. (German and Russian summaries)

The solution is constructed by means of resisting moments introduced along the clamped portion of the edges to cancel the slope of the deflection surface produced by a given load only in such portions. Some numerical results of a square plate under a uniform load or a point load are obtained. Their agreement with experimental data is discussed.

S. C. Das (Madras)

6163:

Agar'ov, V. A. Method of initial functions in the technical theory of bending of rectangular plates. *Dopovidi Akad. Nauk Ukrain. RSR* **1959**, 1206-1210. (Ukrainian. Russian and English summaries)

The deflection surface of a rectangular isotropic plate subjected to arbitrary boundary conditions is obtained. The equation of that deflected surface takes the form of a sum of transcendental differential operators applied to four boundary functions. In the general case boundary conditions at two opposite edges can be satisfied exactly, at two others only approximately. A numerical example, concerning a plate with all edges clamped, is given.

Z. Kączkowski (Warsaw)

6164:

Soare, Mircea. ★Aplicarea ecuațiilor cu diferențe finite la calculul plăcilor curbe subțiri [Application of finite difference equations to the calculation of thin shells]. *Biblioteca Științelor Tehnice*, Vol. VI. Editura Academiei Republicii Populare Române, Bucharest, 1959. 401 pp. (7 inserts)

In this book the method of finite differences is systematically applied to the solution of different problems of thin shells, in the cases when the exact methods of calculation do not lead to an effective result.

In chapter I, the general equations of equilibrium of thin shells are established, giving also some consideration to the solutions of these equations and to simplifying the suppositions customarily used. Then in chapter II, the method of finite differences is analyzed from different points of view as well as the possibility of its application to the solution of thin shells. The membrane theory for thin rotary shells in which there are symmetrical stress states is considered in chapter III. The case of non-symmetrical stress states is considered in chapter IV. In chapter V, the membrane theory for any thin shells is considered. In chapter VI, the membrane theory is applied to the thin shells of translation. Chapter VII is dedicated to the membrane theory for thin shells whose surfaces possess two directrices and a directing plane. The bending theory for thin rotary shells is exhibited in chapter VIII, and, in chapter IX, this theory is applied to the prismatic and cylindrical thin shells. Different types of shells, simply supported or not on tympanons, plates with ribs, prestressed plates, etc., are examined. Chapter X, the last, is concerned with the study of thin shells with the aid of model tests. At the end of the book are given six appendices necessary for the effective calculation of different types of shells. In various chapters numerical examples of calculation are given; many of these were effectively constructed in different industrial buildings.

The reviewer would like to mention that the book, containing an extensive and up-to-date bibliography, contains also very many of the author's own results, especially those concerning the plurilocal method, the calculation of hyperbolic towers, etc., as well as the results of other scientific workers of Rumania. He considers the work a valuable monograph in the important domain of equilibrium of thin elastic shells, which will be of real utility for engineering design.

N. Cristescu (Bucharest)

6165:

Kiltshewskij, N. A. Integrodifferential- und Integralgleichungen für das Gleichgewicht dünner elastischer Schalen. *Z. Angew. Math. Mech.* **40** (1960), 153-161. (English and Russian summaries)

The paper describes a method, based on the reciprocity theorem, for obtaining the integro-differential and integral

equations for the displacements pertaining to the state of equilibrium of thin elastic shells. This method is a modified version of the well-known method due to Somigliana from which it differs by the special choice of a system of "auxiliary displacements". Due to this choice, the displacement of any point on the medium surface of the shell may be expressed as the sum of the displacement of the corresponding point on the medium surface of a plate—called the "image surface" of the medium shell surface—and an additional displacement depending mainly on the curvature of the medium surface of the shell. This method, used by the author 10 years ago for cylindrical shells, is now extended to apply to shells with a medium surface of any shape whatsoever.

W. T. Koiter (Delft)

6166:

Pelka, Zbigniew. Computation of translational shells by means of the method of funicular polygon. *Rozprawy Inż.* 7 (1959), 463-480. (Polish. Russian and English summaries)

Translational shells in the membrane state are studied. The differential equation of the stress function used is replaced by a system of algebraic linear equations. The left-hand side of each equation contains the unknown values of the stress function at 9 neighbouring nodes. The right-hand term involves known values of the load at the same 9 nodes. A numerical example shows that the method of funicular polygon is more accurate than that of finite differences.

Z. Kączkowski (Warsaw)

6167:

Houghton, D. S.; Johns, D. J. Deformation equations for non-circular cylinders. *J. Roy. Aero. Soc.* 64 (1960), 765-766.

6168:

Chao, Hwei-yuan. On the torsion of non-homogeneous anisotropic elastic cylinders. *Sci. Sinica* 9 (1960), 47-61.

The differential equation for the stress function is written down for the case of anisotropic cylinders under torsion. The cylindrical anisotropic medium and composite cylinders are then discussed. The circular shaft consisting of a medium with continuous strain coefficients is also dealt with. Some numerical examples are given.

S. C. Das (Madras)

6169:

Hayashi, Takuo. On the tension in an orthogonally aeolotropic strip with a circular hole. *Bull. JSME* 3 (1960), 265-270.

This paper deals with the plane stress problem of an orthotropic infinite strip  $x = \pm \infty$  with its straight edges  $y = \pm 1$  parallel to the elastic axis of the plate, and having a circular hole of radius  $\lambda$  with its centre at the origin. The strip is under tension at infinity. The stress at any point in the strip is determined in the form of a power series in  $\lambda$  and numerical results are given for an oak strip when the fiber is (i) parallel, and (ii) perpendicular to the straight edges.

R. M. Morris (Cardiff)

6170:

Hayashi, Takuo. On the bending of an orthogonally aeolotropic strip with a circular hole. *Bull. JSME* 3 (1960), 270-274.

In this paper the bending of the orthotropic infinite strip described in the preceding review is treated by the same method. The strip, instead of being under tension at infinity, is under the action of a bending moment in its own plane.

R. M. Morris (Cardiff)

6171:

Hawley, F. J. Note on a paper by G. M. L. Gladwell "Some mixed boundary value problems of aeolotropic thin plate theory". *Quart. J. Mech. Appl. Math.* 13 (1960), 38-39.

The paper referred to is in same J. 12 (1959), 72-81; MR 21 #2404.

6172:

Woinowsky-Krieger, S. Über die Biegung einer unendlich erstreckten orthotropen Platte auf elastischer Unterlage. *Ing.-Arch.* 29 (1960), 22-30.

An orthotropic plate resting on an elastic foundation of arbitrary isotropic properties is considered. The deflected surface of an infinite plate is presented in terms of a double Fourier integral. Some particular cases of loading and that of orthotropy, concerning the plate resting on an elastic foundation of Winklerian type and on an elastic half space, are considered. The numerical tables facilitate finding extrema of deflection, reaction of foundation and bending moments due to a concentrated load.

Z. Kączkowski (Warsaw)

6173:

Vvedenskaya, A. V. The dislocation field in the case of breaks in continuity of an elastic medium. *Izv. Akad. Nauk SSSR. Ser. Geofiz.* 1959, 516-526. (Russian)

The author determines the dynamic field of dislocations in an isotropic and homogeneous elastic space in the presence of breaks in the continuity of the material and with given displacements at the boundary of the discontinuity. The solution is based on Volterra's theory and is bound up with the determination of the stresses which are removed at the boundary of the surface of discontinuity at the instant of its formation. The results can be used to study the stresses and discontinuities at the epicenter of an earthquake. The case of a spherical surface of discontinuity is treated in some detail and the results are illustrated by diagrams.

L. M. Milne-Thomson (Madison, Wis.)

6174:

Indenbom, V. L. Reciprocity theorems and influence functions for the dislocation density tensor and the deformation incompatibility tensor. *Dokl. Akad. Nauk SSSR* 128 (1959), 906-909 (Russian); translated as *Soviet Physics. Dokl.* 4 (1960), 1125-1128.

By generalising the Maxwell-Betti reciprocal theorem in the case of residual deformations it is shown that the stress produced by a concentrated force plays the part of an influence function for such deformations. The idea is then utilised to formulate analogous reciprocal theorems for stresses caused by a dislocation distribution or a

deformation incompatibility distribution. As examples of the application of the theorems, stresses produced by a concentrated force in (i) an unbounded medium and (ii) a semi-infinite medium have been chosen as influence functions to calculate displacement vector due to dislocation loops in these media. It is hoped that various problems of elastic dislocation and incompatibility deformations may conveniently be solved by the method.

G. Paria (Kharagpur)

6175:

Volterra, E.; Morell, J. D. A note on the lowest natural frequency of elastic arcs. *J. Appl. Mech.* 27 (1960), 744-746.

6176:

Ahbe, H. Theoretische Analyse des Tonspektrums der mit einem nachgiebigen Hammer angeschlagenen Saite. *Z. Angew. Math. Mech.* 40 (1960), 289-309. (English and Russian summaries)

From the author's summary: "A well-known integral representation is used to compute the amplitudes of the partial vibrations of a percussed chord. The integral contains the force exerted by the hammer on the chord. Discounting more or less plausible assumptions, this force is not directly known but has to be found as a solution of a Volterra integral equation of the second kind. For the linear case an exact solution, and for the general case a method of approximation is given."

W. E. Boyce (Troy, N.Y.)

6177:

Gladwell, G. M. L. The receptance function of a beam. *Mathematika* 6 (1959), 134-141.

The author discusses the construction of the "receptance function"—defined as the negative of the Green's function—for the transverse vibrations of an elastic beam flexibly supported at each end. The forcing function is assumed periodic in time but may be distributed in space along the beam. The construction is valid when the driving frequency differs from the natural frequencies, and, also provided a certain orthogonality condition is met, when they are equal. No examples are furnished.

W. E. Boyce (Troy, N.Y.)

6178:

McKinney, Wilbur T. Matrix equations for the determination of beam natural frequencies. *J. Soc. Indust. Appl. Math.* 8 (1960), 436-457.

The paper details a technique for setting up vibration equations for a lumped mass representation of a beam in bending, beam shear and torsion, with the effects of mass torsional moment of inertia and rotary inertia included. It is stated that this is incorporated in an extensive digital computer program developed by Convair-Astronautics for automatic derivation, from basis structural data, of beam loading parameters and generalised masses in addition to natural modes and frequencies. The technique uses a single set of influence coefficients (that of the cantilever boundary conditions) and equations for any chosen boundary conditions are derived by imposition of appropriate constraint and/or equilibrium conditions. The reviewer believes that the use of a right-hand coordinate axis

system would have some advantage if extension of the program into overall vehicle dynamics were contemplated.

R. Traill-Nash (Melbourne)

6179:

Capriz, G. On the vibrations of shafts rotating on lubricated bearings. *Ann. Mat. Pura Appl.* (4) 50 (1960), 223-248.

Assuming complete and stable films of lubricant, approximate expressions are derived for the components of force on the rotating shaft due to the lubricant for three limiting cases of bearing geometry. The classical partial differential equations of the vibrations of uniform shafts, subject to the non-linear boundary conditions associated with the above force components, are studied. This leads to criteria for the speeds of rotation at which self-excited vibration or oil whirl occurs.

G. B. Warburton (Edinburgh)

6180:

Chaudhury, B. B.; Chatterjee, P. N. Resonant frequency for longitudinal vibration of a uniform beam with internal damping. *Proc. 4th Congress Theoret. Appl. Mech.* 1958, pp. 293-298. *Indian Soc. Theoret. Appl. Mech.*, Kharagpur.

6181:

Jones, R. P. N.; Mahalingam, S. The natural frequencies of free and constrained non-uniform beams. *J. Roy. Aero Soc.* 64 (1960), 697-699.

6182:

Solecki, Roman. Free and forced vibration of a triangular plate. *Rozprawy Inż.* 8 (1960), 63-81. (Polish. Russian and English summaries)

The problem of free and forced vibrations of an isotropic thin plate representing a half of a regular triangle is considered in the paper. The author derives the equation of eigen-frequency in the following cases: the plate simply supported along its periphery; the plate with additional point supports (numerical examples are given); and the plate clamped at some or at all of its edges. The last cases lead to the Fredholm integral equation of the first kind.

Z. Kączkowski (Warsaw)

6183:

Solecki, Roman. General solution for a plate having the form of a right-angled triangle. *Rozprawy Inż.* 8 (1960), 293-322. (Polish. Russian and English summaries)

An isotropic plate as described in the title, resting on an elastic foundation of Winklerian type, is loaded in its plane by uniformly distributed compressing forces. The plate satisfies at its edges arbitrary but continuous boundary conditions. Using the method of eigen-transform the author obtains the amplitude of free or forced vibrations of the plate. The general solution obtained practically can be used if the eigen-functions are known. For the case of a 45°-45°-90° triangle, when the eigen-functions are known, a more detailed analysis is presented. For ten various cases of support of the plate, the infinite system of linear algebraic equations is derived in the paper. No new eigen-values are given.

Z. Kączkowski (Warsaw)

6184:

Satō, Yasuo; Yamaguchi, Rinzo. Coupling effect of shear vibrations of the structure with elastic foundations, and the maximum response of rocking motion. *Bull. Earthquake Res. Inst. Tokyo* 38 (1960), 369-383. (Japanese summary)

The nature of oscillations of a simple structure on an elastic foundation associated with purely horizontal seismic waves propagating in the foundation is studied. For simplicity the stress within the circular base of the structure is assumed uniform. As solutions of the equations of motion for the structure, expressions for the displacements at different points of the structure associated with a harmonic wave are obtained, and their variation with the material constants as well as with the dimensions of the structure are also presented graphically.

S. K. Chakrabarty (Howrah)

6185:

Hersch, Joseph. Un principe du type de Thomson pour l'équilibre d'une plaque encastrée chargée. *C. R. Acad. Sci. Paris* 250 (1960), 2992-2994.

Let  $G$  be a domain of the  $x, y$  plane, and let  $\Gamma$  be its boundary. The equation studied is  $\Delta \Delta u = p(x, y)$  in  $G$ , with  $u = \partial u / \partial n = 0$  on  $\Gamma$ . The energy integral is  $Q = \iint_G (\Delta u)^2 dA$ . The author obtains upper bounds for  $Q$  by means of a principle related to that of Thomson.

Let the axes  $\xi, \eta$  be obtained by rotating the  $x, y$  axes through  $45^\circ$ . The author describes a class of functions  $f_1, f_2, f_3, f_4$  defined on  $G$  such that

$$Q \leq \frac{2}{3} \iint_G (f_{1xx}^2 + f_{2yy}^2 + f_{3\eta\eta}^2 + f_{4\eta\eta}^2) dA,$$

and this becomes an equality if  $f_1 = f_2 = f_3 = f_4 = u$ .

If  $\alpha, \beta, \gamma$  are three directions forming angles of  $120^\circ$ , then the inequality takes the form

$$Q \leq \frac{8}{9} \iint_G (g_{1\alpha\alpha}^2 + g_{2\beta\beta}^2 + g_{3\gamma\gamma}^2) dA.$$

If the fourth derivatives of the  $f$ 's and  $g$ 's are chosen as constants, particularly simple estimates of  $Q$  are obtained, and these are exact in case  $G$  is a circle or an infinite strip.

W. W. Hooker (Palo Alto, Calif.)

6186:

Hersch, Joseph. Une méthode pour l'évaluation par défaut de la première valeur propre de la vibration ou du flambage des plaques encastrées. *C. R. Acad. Sci. Paris* 250 (1960), 3943-3945.

Let  $G$  be a domain of the  $x, y$  plane, and let  $\Gamma$  be its boundary. The equation of the vibrating clamped plate is  $\Delta \Delta u - \lambda_1 u = 0$  in  $G$ , with  $u = \partial u / \partial n = 0$  on  $\Gamma$ . Let the axes  $\xi, \eta$  be obtained by rotating the  $x, y$  axes through  $45^\circ$ . For a simple class of functions  $f_1, f_2, f_3, f_4$  defined on  $G$ , the author obtains the lower bound

$$\lambda_1 \geq \frac{2}{3} \inf_G (f_{1xxxx}/f_1 + f_{2yyyy}/f_2 + f_{3\eta\eta\eta\eta}/f_3 + f_{4\eta\eta\eta\eta}/f_4).$$

Similarly, if  $\alpha, \beta, \gamma$  are three fixed directions forming angles of  $120^\circ$ , then

$$\lambda_1 \geq \frac{8}{9} \inf_G (g_{1\alpha\alpha\alpha\alpha}/g_1 + g_{2\beta\beta\beta\beta}/g_2 + g_{3\gamma\gamma\gamma\gamma}/g_3),$$

where the  $g_i$  satisfy conditions similar to those for the  $f_i$ .

The problem of the buckling clamped plate is  $\Delta \Delta U + \Lambda_1 \Delta U = 0$  in  $G$ , with  $U = \partial U / \partial n = 0$  on  $\Gamma$ . Conditions on  $f_i$  and  $g_i$  are given such that

$$\Lambda_1 \geq \frac{1}{3} \inf_G [f_1 + f_2 + f_3 + f_4 - \sqrt{((f_1 - f_2)^2 + (f_3 - f_4)^2)}],$$

$$\Lambda_1 \geq \frac{4}{9} \inf_G [\bar{g}_1 + \bar{g}_2 + \bar{g}_3$$

$$- \frac{1}{\sqrt{2}} \sqrt{((\bar{g}_1 - \bar{g}_2)^2 + (\bar{g}_2 - \bar{g}_3)^2 + (\bar{g}_3 - \bar{g}_1)^2)}]$$

where  $\bar{f}_1 = -f_{1xxxx}/f_{1xx}$ , etc.

Particularly simple estimates result in the two problems by choosing  $f_{1xxxx}/f_1$  and  $\bar{f}_1$  to be independent of  $x$  (and similarly for the other directions).

W. W. Hooker (Palo Alto, Calif.)

6187:

Lin, Y. K. Free vibration of continuous skin-stringer panels. *J. Appl. Mech.* 27 (1960), 669-676.

Author's summary: "The determination of the natural frequencies and normal modes of vibration for continuous panels, representing more or less typical fuselage skin-panel construction for modern airplanes, is discussed in this paper. The time-dependent boundary conditions at the supporting stringers are considered. A numerical example is presented, and analytical results for a particular structural configuration agree favorably with available experimental measurements."

6188:

Mindlin, R. D. Waves and vibrations in isotropic, elastic plates. *Structural mechanics*, pp. 199-232. Pergamon Press, New York, 1960.

The formal solution of wave propagation in elastic plates goes back to the end of the last century; the full implications of the theory have only been elucidated during the past few years, however, and the manner in which high frequency vibrations in bounded plates can be treated has now become clear. Much of the credit for this clarification goes to the work of the author who in the present expository review first treats the propagation of waves in an elastic half-space, then considers waves in an infinite plate and then shows the problems confronted in treating a bounded plate where a discussion of approximate methods is given. H. Kolsky (Providence, R.I.)

6189:

Goldberg, John E.; Bogdanoff, John L.; Marcus, Lee. On the calculation of the axisymmetric modes and frequencies of conical shells. *J. Acoust. Soc. Amer.* 32 (1960), 738-742.

On the grounds that axisymmetric modes are the important ones in loudspeaker cone design, the equations of free harmonic vibration of a differential element of cone are set up using ordinary thin shell theory. These are six first-order differential equations in the amplitudes of displacements and forces. Six linear transformations in terms of the amplitudes of the bending moment, shear and membrane force at the small end of the cone transform the two-point boundary-value problem into an initial value problem. The equations are then integrated by the Runge-Kutta fourth-order process using fifteen

equal intervals along the generator. A trial-and-error procedure is used in which assumed frequencies are adjusted until terminal boundary conditions are satisfied. The mode shapes and frequencies computed for the first three modes of a cone are shown.

W. W. Soroka (Berkeley, Calif.)

6190:

Franken, Peter A. Input impedances of simple cylindrical structures. *J. Acoust. Soc. Amer.* **32** (1960), 473-477.

A harmonic radial point force on an infinitely long thin circular-cylindrical shell and the consequent surface displacement components in vacuo are expanded in Fourier sum integrals, substituted into membrane equations for displacements, and Fourier amplitude factors for radial motion obtained for frequencies below the characteristic radial resonance. Fourier inversion by complex contour integration yields the radial displacements, from which an expression for the modal driving point impedance is obtained. A "resonance factor" and damping are introduced on physical grounds. Special consideration is given to axisymmetric motion. A simply-supported finite cylinder is considered as well as uniform distributions of radial driving forces around a circumference and along a surface element. The origin of the driving force is related to the motion of an elastically-mounted mass.

W. W. Soroka (Berkeley, Calif.)

6191:

Lin, Y. K. Coupled bending and torsional vibrations of restrained thin-walled beams. *J. Appl. Mech.* **27** (1960), 739-740.

6192:

Lin, Y. K.; Lee, F. A. Vibrations of thin paraboloidal shells of revolution. *J. Appl. Mech.* **27** (1960), 743-744.

6193:

Teleman, Silviu. On the first problem of self vibration of an elastic body. *Rev. Math. Pures Appl.* **4** (1959), 665-684.

The author employs the methods of spectral analysis to study the free vibrations of a homogeneous, isotropic, bounded, elastic body whose boundary is fixed. This is a continuation of the author's earlier work [*Acad. R. P. Romine. Bul. Sti. Sect. Sti. Mat. Fiz.* **7** (1955), 105-125; *MR* **17**, 684] and also extends results of Friedrichs [*Ann. of Math.* (2) **48** (1947), 441-471; *MR* **9**, 255] to regions with more general boundaries. Among other things the author discusses the analyticity of solutions and the convergence of approximate solutions.

W. E. Boyce (Troy, N.Y.)

6194:

Arnold, Lee. Aeroelasticity. *Frontiers of numerical mathematics*, pp. 59-68. University of Wisconsin Press, Madison, Wis., 1960.

This is a brief review of the problem of determining the aerodynamic forces on oscillating wings in compressible flow. Subsonic, transonic, supersonic, and hypersonic situations are touched upon, and some of the mathematical features of each are pointed out.

W. R. Sears (Ithaca, N.Y.)

6195:

Bobesko, A.; Kacprzyński, J.; Kaliski, S. Vibration and elastic stability of slender bodies in linearized supersonic flow. *Proc. Vibration Problems No. 4* (1960), 77-89. (Polish and Russian summaries)

Slender-body and beam approximations are used to obtain the equation of motion of a flexible body of revolution. Assuming harmonic motion and representing the cross section and other beam parameters as polynomials in the axial coordinate ( $x$ ), the eigenvalue problem is attacked by Laplace-transforming with respect to  $x$  and then expressing the inverse transform as a Volterra integral equation.

J. W. Miles (Los Angeles, Calif.)

6196:

Jarmai, L.; Szereday, E. Buckling of connected parallel beams. *Les mathématiques de l'ingénieur*, pp. 348-355. *Mém. Publ. Soc. Sci. Arts Lett. Hainaut*, vol. hors série, 1958.

The presentation is difficult to understand, but the system studied is apparently one of  $m$  vertical columns with ends at the same level, constrained to the same buckling deflections at  $n$  stations which are also at the same levels. There is an arbitrary distribution of longitudinal loads and of stiffnesses, except that the stiffnesses are constant between stations. The author sets up a solution in matrix form, and gives several examples. He ignores the obvious elementary solution of replacing the system with one column with lumped loads and stiffnesses; this should give an excellent approximation if the stations are fairly numerous and well spaced.

L. H. Donnell (Ann Arbor, Mich.)

6197:

Born, J. Der biegesteife Kreisring unter periodischer Belastung. *Österreich. Ing. Z.* **3** (1960), 204-206.

6198:

Singer, J. The effect of axial constraint on the instability of thin circular cylindrical shells under external pressure. *J. Appl. Mech.* **27** (1960), 737-739.

6199:

Thielmann, W.; Schnell, W.; Fischer, G. Beul- und Nachbeulverhalten orthotroper Kreiszyklinderschalen unter Axial- und Innendruck. *Z. Flugwiss.* **8** (1960), 284-293. (English and French summaries)

Authors' summary: "The buckling and post-buckling behaviour of isotropic and orthotropic circular cylinders being under axial compression and internal pressure is determined by means of an approximate large-deflection theory."

6200:

Gregory, M. The bending and shortening effect of pure torque. *Austral. J. Appl. Sci.* **11** (1960), 209-216.

Author's summary: "The general analysis of the bending and shortening of thin open-section members under pure torque is developed, and experimental work is given in confirmation."

6201:

Gregory, M. The application of the Southwell plot on strains to problems of elastic instability of framed structures, where buckling of members in torsion and flexure occurs. *Austral. J. Appl. Sci.* 11 (1960), 49-64.

6202:

Hill, R.; Sewell, M. J. A general theory of inelastic column failure. I, II. *J. Mech. Phys. Solids* 8 (1960), 105-111, 112-118.

Hill's theory of uniqueness and stability of inelastic solids [same *J.* 7 (1959), 209-225; MR 21 #3978] is applied to column theory. Critical condition for bifurcation (non-uniqueness) is obtained, generalizing Shanley's tangent-modulus formula and allowing for shear deformation. From sufficient conditions for stability it appears that critical load for stability is higher than bifurcation load. It corresponds to Engesser's and Kármán's reduced modulus value, modified to allow for the effect of shear deformation.

W. T. Koiter (Delft)

6203:

Hult, Jan. Creep buckling of plane frameworks. *Kungl. Tekn. Högsk. Handl. Stockholm*. No. 136, 31 pp. (1959).

The author discusses linear viscoelastic creep buckling in frameworks in analogy to elastic buckling theory. Non-linear creep buckling in frameworks with idealized cross-sections of members is studied by means of assumption of concentrated creep hinges. The essential complication of creep buckling vs. elastic buckling {in case of linearized theory (addition by reviewer)} is that variables, such as bending moments and deflections, do not show a constant ratio during the buckling process.

W. T. Koiter (Delft)

6204:

Wallis, Richard F. Theory of surface modes of vibration in two- and three-dimensional crystal lattices. *Phys. Rev.* (2) 116 (1959), 302-308.

From author's summary: "Theoretical expressions have been developed for the frequencies and displacements of the normal modes of vibration for two- and three-dimensional alternating diatomic lattices with free boundaries. Only square and cubic lattices are considered. Nearest-neighbor Hooke's law forces having both longitudinal and transverse components are assumed. The results have been obtained both by a perturbation method in which the ratio of the transverse and longitudinal force constants is treated as a small quantity and by a Green's function method. The use of the free boundary condition leads to the existence of surface modes of vibration in which the displacement amplitude is relatively large for a light atom on a boundary and decreases roughly exponentially toward the interior of the lattice. A band of surface mode frequencies lies in the 'forbidden' gap between the acoustical and optical branches."

W. E. Boyce (Troy, N.Y.)

6205:

Mitra, M. On the solution of problems of dynamic plane elasticity for anisotropic media. *Quart. J. Mech. Appl. Math.* 13 (1960), 369-373.

A complex variable method, based on the work of

Radok [*Quart. Appl. Math.* 14 (1956), 289-298; MR 18, 349], is used to obtain a general solution of problems involving the steady motion of forces applied to the surface of a semi-infinite, anisotropic half space under conditions of plane strain. The method is specialised to the cases of a moving parabolic punch and a moving dislocation.

G. Eason (Newcastle-upon-Tyne)

6206:

Gvozdev, A. A. On the conditions at elastic wave fronts propagating in a nonhomogeneous medium. *Prikl. Mat. Meh.* 23 (1959), 395-397 (Russian); translated as *J. Appl. Math. Mech.* 23, 556-561.

The author states results obtained for the properties of elastic waves in non-homogeneous media in which the displacement vector and its derivatives are discontinuous along the wave front. The theory is developed by use of a weak solution of the equations of motion involving an integration over a four-dimensional region of Euclidean space-time and Gauss' theorem. Previous authors have discussed the corresponding theory for the homogeneous case [M. L. Levin and S. M. Rytov, *Akust. Ž.* 2 (1956), 173-176; MR 18, 431; V. M. Babič and A. S. Alekseev, *Izv. Akad. Nauk SSSR. Ser. Geofiz.* 1958, 17-31; MR 20 #516; G. A. Skuridin and A. A. Gvozdev, see following review]. A typical result is the following. If  $u$  is the discontinuity of the displacement vector,  $a$  is the speed of propagation of the longitudinal wave,  $\lambda$  and  $\mu$  are the elastic "constants", and  $\rho$  is the density, then

$$\text{curl } u = \frac{u \times P}{\lambda + \mu}, \quad P = a^2 \text{ grad } \rho - 2 \text{ grad } \mu.$$

(For the case of compressible fluids, the reviewer introduced an integral formulation of the equations of motion by use of integrals over closed hypersurfaces of space-time; see N. Coburn, *Math. Mag.* 27 (1954), 245-264 [MR 15, 1000]. This approach obviates the use of Gauss' theorem when the velocity vector is discontinuous. A similar approach would be desirable for the displacement vector in elasticity.)

N. Coburn (Ann Arbor, Mich.)

6207:

Skuridin, G. A.; Gvozdev, A. A. Boundary conditions for the jumps of discontinuous solutions of the dynamical equations of the theory of elasticity. *Izv. Akad. Nauk SSSR. Ser. Geofiz.* 1958, 145-156. (Russian)

The authors study the time-dependent linear equations of elasticity for an isotropic non-homogeneous medium in order to determine the conditions satisfied by the jump of the deformation vector along a moving discontinuity (which we shall call a shock) manifold in Euclidean three-space. In particular, it is assumed that the jump vector is an elementary wave of exponential type which satisfies the elasticity equations. The problem is reduced to the determination of two functions, each of which satisfies a linear differential equation, with coefficients dependent on the two known wave speeds. Hence, the solution can be determined explicitly. Application is made to the case of three plane shocks which meet at a point. Finally, a more general case of plane shocks is studied. (Reviewer's note: a similar study of the case of finite deformations may show whether the assumption of small deformations along the shock wave is permissible.)

N. Coburn (Ann Arbor, Mich.)

6208:

Skuridin, G. A. Duhamel's principle and asymptotic solutions of dynamical equations in the theory of elasticity. II. *Izv. Akad. Nauk SSSR. Ser. Geofiz.* 1959, 337-343. (Russian)

In Part I [same *Izv.* 1959, 3-10; MR 21 #2417] an asymptotic expansion for the displacement vector in an isotropic elastic medium was derived. The coefficients in this expansion were expressed in terms of the discontinuities  $[\partial u / \partial t^{\nu}]$ ,  $\nu = 0, 1, 2, \dots$ , in the time derivatives of a discontinuous "pulse" solution. In this paper, the author derives the first member of an infinite recursive system of ordinary differential equations satisfied by the discontinuities  $[\partial u / \partial t^{\nu}]$  and refers the reader to two earlier papers [see preceding review and Skuridin, *ibid.* 1956, 625-633; MR 19, 1215], for derivation of the complete system and its boundary conditions.

R. A. Toupin (Washington, D.C.)

6209:

McClintock, F. A.; Sukhatme, S. P. Travelling cracks in elastic materials under longitudinal shear. *J. Mech. Phys. Solids* 8 (1960), 187-193.

6210:

Broberg, K. Bertram. The propagation of a brittle crack. *Ark. Fys.* 18, 159-192 (1960).

The main part of this paper is a solution, by use of a Laplace transform, of a two-dimensional elasto-dynamic mixed boundary problem for a half-space, with given normal displacement outside the strip. The parallel edges of the strip separate from zero with uniform velocity. The solution is elegant, well-presented, and valuable as a contribution to the literature of elasticity.

In discussing the connection between this solution and the propagation of a crack, however, the author uses neither the energy criterion of Griffith-Irwin theory nor the (equivalent) theory of cohesion modulus put forward by Barenblatt. Instead he postulates finite stress at the ends of a crack and is thereby led to the deduction that the velocity of the crack is that of a Rayleigh wave.

J. W. Craggs (Newcastle-upon-Tyne)

6211:

Kurlandzki, Jerzy. On certain plane diffraction and reflection problems of elastic waves. *Proc. Vibration Problems No. 3* (1960), 35-51. (Polish and Russian summaries)

This paper is concerned with the diffraction and reflection effects which arise when a disturbance, travelling in a homogeneous, isotropic elastic medium, encounters a free surface or a contact discontinuity. The analysis is expressed in terms of cylindrical polar coordinates  $r, \phi, z$ , it being assumed throughout that conditions of plane strain hold in planes  $z = \text{const}$ .

Two groups of problems are discussed. The first group deals with the diffraction of a disturbance about the cylindrical cavity bounded by the surface  $r = r_0$ , the following cases being considered: (1)  $r = r_0$  is a free surface. (2) The cavity is filled with elastic material the properties of which differ from those of the surrounding medium. The two solids may be in welded contact at  $r = r_0$  or may be able to exert normal but not tangential stresses on one another. (3) The cavity wall is reinforced by a

cylinder of different material occupying the region  $r_1 \leq r \leq r_0$ . The inner face of the cylinder is a free surface and the interface may be either a welded or a frictionless contact as in (2).

The second group consists of the following problems: (1) The reflection of a disturbance incident upon the wedge-shaped free surface formed by the planes  $\phi = 0, \phi = \phi_0 (> \pi)$ . (2) The reflection and diffraction of a disturbance incident upon the free surface formed by the planes  $\phi = 0, \phi = \phi_0$  and the cylinder  $r = a$ .

Each problem is solved for a general incident disturbance, but some of the results are specialized to the case in which the incident wave has harmonic time dependence. Extensive use is made of integral transform methods and the solutions are derived in a purely formal manner. The author recognizes, however, the need to establish the existence of solutions and to verify that the integral expressions obtained formally in fact satisfy the appropriate differential systems. The justification of the first group of solutions is briefly examined. The inversion of the transforms is not discussed and no results are given for specific incident disturbances. P. Chadwick (Sheffield)

6212:

Serbin, Hyman. Intense stress field produced in a semi-infinite elastic solid by a bomb blast at the surface. *J. Acoust. Soc. Amer.* 32 (1960), 1250-1256.

Author's summary: "The response of a solid subjected to intense blast pressures depends on the ratio of the applied pressures to the stresses (yield stress, etc.) characteristic of the material of the solid. When this ratio is small, the solid can be expected to respond elastically. The paper develops the elastic theory of a semi-infinite solid exposed to a blast initiated at a point on the surface, assumed to be in air at sea level. The results are explicit expressions for the displacements and stresses for early time, small radial distances, and shallow depths. Application to the case where the solid represents the earth is discussed." L. S. D. Morley (Farnborough)

6213:

Serbin, Hyman. Propagation of intense shock waves into the earth. *J. Acoust. Soc. Amer.* 32 (1960), 1257-1262.

Author's summary: "An earlier paper [#6212] by the author considered the behaviour of a semi-infinite solid reacting elastically to a point-initiated blast wave at the surface. When the applied pressures are sufficiently high, the solid will experience inelastic behaviour characteristic of the material. The earth is here regarded as a medium made up of solid matter, water, and air voids. It is assumed that the principal effect of high normal stresses is to close up the air voids and that thereafter the medium acts like an elastic solid. The calculation of the peak pressure vs. depth is compared with experimental data derived from nuclear bomb tests and shows reasonable agreement considering the wide scatter of test points." L. S. D. Morley (Farnborough)

6214:

Miles, John W. Low-frequency motion of bodies in an elastic wave field. *J. Acoust. Soc. Amer.* 32 (1960), 1396-1401.

The author derives approximate expressions for the motion (mean translation and rotation) of an elastic body embedded in another slightly different (in constants, density) linear, homogeneous, elastic solid. An arbitrary incident harmonic wave field is considered, and it is assumed that the characteristic length of the body is small compared to the wavelengths involved.

The author obtains the general solution to the problem by reducing it to the evaluation of two integrals in potential theory. These integrals are evaluated for the ellipsoid. Results are given for the special cases of the sphere and prolate spheroid subjected to incident plane *P*- or *S*-wave excitation.

*J. Miklowitz (Pasadena, Calif.)*

6215:

**Fredricks, R. W.; Knopoff, L.** The reflection of Rayleigh waves by a high impedance obstacle on a half-space. *Geophysics* **25** (1960), 1195-1202.

Authors' summary: "The reflection of a time-harmonic Rayleigh wave by a high impedance obstacle in shearless contact with an elastic half-space of lower impedance is examined theoretically. The potentials are found by a function-theoretic solution to dual integral equations. From these potentials, a 'reflection coefficient' is defined for the surface vertical displacement in the Rayleigh wave. Results show that the reflected wave is  $\pi/2$  radians out of phase with the incident wave for arbitrary Poisson's ratio. The modulus of the 'reflection coefficient' depends upon Poisson's ratio, and is evaluated as  $r_R = 0.265$  for  $\sigma = 0.25$ ."

6216:

**Walters, K.** The motion of an elastico-viscous liquid contained between concentric spheres. *Quart. J. Mech. Appl. Math.* **13** (1960), 325-333.

The author uses linear viscoelasticity theory to analyze the flow between two concentric spheres when the outer one undergoes forced angular oscillations and the inner one is constrained by a torsional wire. He discusses theoretical advantages of such a viscometer, compared to Couette viscometers. *J. L. Ericksen (Baltimore, Md.)*

6217:

**Nettleton, R. E.** Thermodynamics of viscoelasticity in liquids. *Phys. Fluids* **2** (1959), 256-263.

This is a physical paper, containing remarks on various structural interpretations for the moduli in the Maxwell linearized visco-elastic theory, with particular reference to infinitesimal waves. *C. Truesdell (Basel)*

6218:

**Hunter, S. C.** Viscoelastic waves. *Progress in solid mechanics*, Vol. 1, pp. 1-57. North-Holland Publishing Co., Amsterdam, 1960.

This paper reviews the theory of wave propagation in linear viscoelastic solids and describes some of the experimental work that has been carried out in this field. The first part of the paper summarizes the mathematical description of linear viscoelastic behavior in terms of creep functions, relaxation functions, relaxation spectra, etc. The use of Laplace transforms and Fourier transforms in solving specific problems is then treated. The second

part of the paper discusses the propagation of uniaxial stress pulses and shows how these may be treated by transform methods. The next section deals with experimental results of wave propagation in high polymers, and the final section is concerned with the general equations of a linear isotropic viscoelastic solid and includes the theory of wave propagation in three dimensions. The paper gives a comprehensive list of references to work carried out in this field. *H. Kolsky (Providence, R.I.)*

6219:

**Klyušnikov, V. D.** New concepts in plasticity and deformation theory. *Prikl. Mat. Meh.* **23** (1959), 722-731 (Russian); translated as *J. Appl. Math. Mech.* **23**, 1030-1042.

The author discusses and compares predictions of (a) the slip theory of Batdorf and Budiansky, (b) the theory of Sanders, (c) his own theory, and (d) an illustrative model of Rabotnov. Theories (a) and (b) and model (d) predict the occurrence of corners in the yield surface at the loading point; theory (c) postulates such corners, but leaves their precise geometry unspecified. The author concludes correctly that (b) can be considered a special case of (c), and that then both theories agree with Nadai's deformation theory for "total loading" in the Sanders sense (stress paths that proceed into the forward cone of tangents to the initial yield surface). But some of his similar speculations concerning theory (a) (based on a simplified 2-dimensional version of the slip theory) are not generally valid. Nevertheless, the author's final general comment is worth quoting: "Quite instructional appears to be the fact that the old deformation theory, whose shortcomings under the conditions of the smoothness of the yield surface made it physically unreliable and thus had a considerable influence on the development of new approaches, precisely from the point of view of new concepts finds its place in the system of general relationships of plasticity." The same point of view underlies the reviewer's closely related study [*J. Appl. Mech.* **26** (1959), 259-264; *MR* **21** #2421].

*B. Budiansky (Cambridge, Mass.)*

6220:

**Ivlev, D. D.** The equations of linearized space problems in the theory of ideal plasticity. *Dokl. Akad. Nauk SSSR* **130** (1960), 1232-1235 (Russian); translated as *Soviet Physics. Dokl.* **5**, 168-171.

First-order perturbation equations for stresses and velocities are formulated for a Lévy-Mises rigid/plastic solid in a three-dimensional state of stress differing infinitesimally from uniform uniaxial tension. The solution is shown to be reducible to two simple wave equations. As a particular problem the initial flow of a slightly grooved tension specimen is considered, though very sketchily; in fact only to the extent of obtaining the boundary conditions on the free surface. Conditions determining the position of the interface between rigid and deforming regions are not investigated. *R. Hill (Nottingham)*

6221:

**Ivlev, D. D.** On the theory of ideally plastic anisotropy. *Prikl. Mat. Meh.* **23** (1959), 1107-1114 (Russian); translated as *J. Appl. Math. Mech.* **23**, 1582-1592.

A yield function is proposed for an anisotropic plastic solid by generalizing Tresca's criterion. In principal stress space the yield surface is a cylinder whose section is an irregular hexagon. Attention is mainly focussed on states of stress corresponding to the vertices of the hexagon, and for these an associated flow rule is formulated. Characteristics for a stress field of this special type are investigated. Plane strain, plane stress, and torsion are discussed in more generality. A somewhat related treatment of anisotropy in plastic torsion, not referred to, was given by R. Hill [*J. Mech. Phys. Solids* 2 (1954), 87-91; MR 15, 1006].  
R. Hill (Nottingham)

6222:

Ivlev, D. D. On the properties of the relations of the law of anisotropic hardening of plastic material. *Prikl. Mat. Meh.* 24 (1960), 144-146 (Russian); translated as *J. Appl. Math. Mech.* 24, 191-194.

Relations along the characteristics are obtained for plane strain of a rigid/plastic solid with an anisotropic yield function proposed by R. T. Shield and H. Ziegler [*Z. Angew. Math. Phys.* 9a (1958), 260-276; MR 20 #6859]. The similar three-dimensional problem for special states of stress corresponding to edges of the yield surface is also remarked on, but too briefly to be comprehensible.

R. Hill (Nottingham)

6223:

Radhakrishnan, S. Plastic buckling of long, thin cylinders under lateral pressure. *Proc. 4th Congress Theoret. Appl. Mech.* 1958, pp. 45-52. Indian Soc. Theoret. Appl. Mech., Kharagpur.

The author studies the title problem, assuming all material to be plastic and ignoring possible strain reversal. He obtains the same result as for the elastic case with Poisson's ratio taken as 0.5 and the modulus of elasticity replaced by  $(E_s + 3E_t)/4$ , where  $E_s$  and  $E_t$  are the secant and tangent moduli. He takes what this reviewer considers to be the unnecessary trouble of starting and following through the general shell theory until the neglects made finally reduce the solution to the quite simple and computationally linear one with all stresses zero except the circumferential, and these constant in the axial direction. Finally, after pointing out correctly that the Donnell equation approximations do not apply to this case since it involves only two circumferential waves, he tries them anyway and finds the expected 4 to 3 over-estimation of strength, as in the elastic case.

L. H. Donnell (Ann Arbor, Mich.)

6224:

Serensen, S. V.; Schneiderovitch, R. M. On the plasticity function under alternating stresses in stress analysis. *Proc. 4th Congress Theoret. Appl. Mech.* 1958, pp. 7-18. Indian Soc. Theoret. Appl. Mech., Kharagpur.

The authors investigate repeated alternating elastic-plastic loading experimentally in terms of torsion of a thin-walled aluminium alloy cylinder. Two types of tests are considered. In a "Symmetric strain cycle" the strain is alternated between equal positive and negative values, whereas in a "symmetric stress cycle" the stress is alternated between equal positive and negative values. In

either case it is found that the stress-strain curve may vary considerably with the cycle number for the first few cycles, but that it rapidly (order of a dozen cycles) achieves a shape essentially independent of the cycle number.

P. G. Hodge, Jr. (Chicago, Ill.)

6225:

Hodge, P. G., Jr.; Sankaranarayanan, R. Plastic interaction curves for annular plates in tension and bending. *J. Mech. Phys. Solids* 8 (1960), 153-163.

A circular plate of uniform thickness with a concentric cut-out under uniform normal load over its surface and uniform tension around its circumference is discussed in the case of perfectly plastic material obeying Tresca's yield condition. The basic nonlinear equations in terms of generalized stresses appropriate to the problem are obtained and the upper and lower bound theorems of limit analysis developed. Close bounds on the plastic interaction curves are found under various support conditions at the inner edge. The results are summarized in tabular form and typical cases are shown graphically.

S. C. Das (Madras)

6226:

Klyušnikov, V. D. On a possible manner of establishing the plasticity relations. *Prikl. Mat. Meh.* 23 (1959), 282-291 (Russian); translated as *J. Appl. Math. Mech.* 23, 405-418.

The author investigates stress-strain relations for strain-hardening materials on the basis of well-defined assumptions for yield surface. Discussion is restricted to loading path in two-dimensional plane of nine-dimensional stress space. Corners in yield surface are included. An important result of the investigation is that within certain limits deformation theory of plasticity is compatible with flow theory.

W. T. Koiter (Delft)

6227:

Lianis, George. The plastic yielding of double notched bars due to pure bending. *Ing.-Arch.* 29 (1960), 55-72.

The bars are of rigid/plastic non-hardening material and are bent in plane strain. A. P. Green [*Quart. J. Mech. Appl. Math.* 6 (1953), 223-239; MR 14, 1041] has proposed associated slip-line and velocity fields in the deforming zone at the initial yield point. The corresponding bending moment could be justified as the actual solution by showing the existence of an equilibrium state of stress, not violating the yield condition, in the remaining rigid part of the bar. On general grounds, however, there is no reason to doubt that Green's values are correct. Nevertheless, the author has considered it worthwhile to obtain lower bounds to the bending moment by using standard (but laborious) methods of constructing so-called statically admissible fields of stress in the entire bar. He has done this for notches which are rectangular, trapezoidal, circular, or vee-shaped with circular fillets. The exercise seems rather pointless since the bounds are not at all close to Green's values and consequently give little indication as to their accuracy. On the other hand, the author reports valuable experimental results obtained with test-pieces of work-hardened copper, which agree very closely with Green's values.

R. Hill (Nottingham)

6228:

Hwang, Chintsun. Plastic bending of a work-hardening circular plate with clamped edge. *J. Aerospace Sci.* 27 (1960), 815-820, 840.

The title problem is formulated as two first order differential equations for the moments  $M_r$  and  $M_\theta$  based on equilibrium and compatibility. The equilibrium equation is linear and is the same for loading or unloading of the material. The compatibility equation is linear for loading but assumes a complex nonlinear form for unloading. Tests are given for both loading and unloading at each step. At the center of the plate  $M_r = M_\theta$  and it is shown that  $M_r = 2M_\theta$  at the clamped edge.

The above stated problem is solved for a load which is uniform spacewise and increase monotonically to half again its yield-point value. Computations were carried out on the NCB 304 Digital Computer and are presented as tables and curves. *P. G. Hodge, Jr. (Chicago, Ill.)*

6229:

Sawczuk, Antoni; Rychlewski, Jan. On yield surfaces for plastic shells. *Arch. Mech. Stos.* 12 (1960), 29-53. (Polish and Russian summaries)

It is assumed that the shell material satisfies the Huber-Mises yield condition and associated flow law. Integrating  $\sigma_{\theta\theta}$  and  $\sigma_{\phi\phi}$  through the shell thickness and assuming that normals to the middle surface remain straight and normal, the authors obtain the six stress resultants  $N_{\theta\theta}$ ,  $M_{\theta\theta}$  as homogeneous functions of the six extension and curvature rates  $\dot{\lambda}_{\theta\theta}$ ,  $\dot{\kappa}_{\theta\theta}$  of the middle surface. Since, for a perfectly plastic material, the magnitude of the strain rates is indeterminate, only five of the generalized strain rates  $\dot{\lambda}_{\theta\theta}$ ,  $\dot{\kappa}_{\theta\theta}$  are independent, the resulting expressions for the generalized stresses  $N_{\theta\theta}$ ,  $M_{\theta\theta}$  constitute a type of parametric representation of a 5-dimensional yield hypersurface in the 6-dimensional generalized stress space. Further, it follows from the plastic flow law for the material that at a plastic section of the shell the generalized strain-rate vector is normal to this hypersurface.

In problems where one or more of the generalized stresses is known to vanish identically, a generalized stress space of fewer dimensions is adequate for a description of the problem. It is shown that the yield hypersurface in such a space of reduced dimensions is obtained by cutting the original hypersurface by an appropriate hyperplane. In this case the corresponding reduced strain-rate vector is again normal to the reduced hypersurface, but the eliminated component of the vector is not generally zero. Alternatively, if one or more generalized strain rates is identically zero, the reduced hypersurface is obtained by projecting the original hypersurface on an appropriate hyperplane. Normality is again preserved but now the eliminated generalized stress component does not generally vanish. The operations of cutting and projecting can coincide if and only if the original hypersurface is symmetric with respect to the eliminated component.

Some specific examples of reduced hypersurfaces are given. Corresponding results are also presented for idealized sandwich shells. *P. G. Hodge, Jr. (Chicago, Ill.)*

6230:

Mróz, Zenon. The load-carrying capacity of orthotropic shells. *Arch. Mech. Stos.* 12 (1960), 85-107. (Polish and Russian summaries)

A theoretical analysis is carried out which is aimed at application to reinforced concrete. It is assumed that the shell is rotationally symmetric and subjected to rotationally symmetric loads. The concrete can take only compressive stresses in either principal direction, and the reinforcing rods in each principal direction are so arranged that they take only tensile stresses. Under these assumptions there is no interaction in the plastic behavior between the two principal directions, so that the yield hypersurface in a four dimensional stress resultant space will be the intersection of two cylinders of the form  $f_1(M_\theta, N_\theta) = 0$ ,  $f_2(M_\phi, N_\phi) = 0$ . A one parameter family of parabolic cylinders  $f_\theta$  are obtained, the parameter being the ratio of the tensile yield strength of the bars to the compressive yield strength of the concrete. The plastic potential flow law is shown to apply for the derived yield surface.

Several examples are solved, the parabolic yield cylinders being first replaced by approximate hexagon yield prisms. The examples include a simply supported spherical cap, a conical frustrum, and a circular cylindrical shell.

*P. G. Hodge, Jr. (Chicago, Ill.)*

6231:

Odqvist, Folke K. G. Non-steady membrane creep of circular plates. *Ark. Fys.* 16, 527-531 (1960).

Incipient creep of metallic membrane is investigated by method similar to earlier paper by author for stationary creep of membrane [*Ark. Fys.* 16 (1959), 113-118; MR 21 #7658]. *W. T. Koiter (Delft)*

6232:

Born, J. Zur Berechnung der Formänderungen aus Kriechen und Schwinden nach dem Verfahren von Busemann. *Österreich. Ing. Z.* 3 (1960), 382-387.

6233:

Kammash, T. B.; Murch, S. A.; Naghdi, P. M. The elastic-plastic cylinder subjected to radially distributed heat source, lateral pressure and axial force with application to nuclear reactor fuel elements. *J. Mech. Phys. Solids* 8 (1960), 1-25.

In this paper the effect of temperature is included only in the generalized Hooke's law—the physical constants are assumed to be temperature independent. The problem is more complicated than that considered by this reviewer in same *J.* 4 (1956), 209-229 [MR 18, 164] and by previous authors in that the axial stress is no longer always the intermediate principal stress. Use of the Tresca yield criterion now gives rise to three different plastic regimes. An analytic solution is obtained for a heat source distribution proportional to  $\exp(-\mu r^2)$  and for a material which workhardens according to the law  $2\dot{\epsilon} = \eta \dot{W}_p$ , where  $\mu$  and  $\eta$  are constants and  $r$  is the radial coordinate. Numerical results are given. *D. R. Bland (Manchester)*

6234:

Kleman, P. W. Survey of thermal problems as affecting the structures of high speed aircraft. *Proc. 4th Congress Theoret. Appl. Mech.* 1958, pp. 151-186. *Indian Soc. Theoret. Appl. Mech.*, Kharagpur.

Author's summary: "This report summarizes the important problems arising from the aerodynamic heating

of high speed aircraft. Problems in the following fields are considered: (1) Aerodynamic heat transfer, (2) heat transfer in structure, (3) thermal stresses, (4) flutter, (5) creep, (6) fatigue, (7) experimental methods, and (8) materials. An extensive bibliography is included."

6235:

Piechocki, Wladyslaw; Ignaczak, Józef. Some problems of dynamic distortion in thermoelasticity. *Arch. Mech. Stos.* **12** (1960), 259-278. (Polish and Russian summaries)

6236:

Mossakowska, Zofia. One-dimensional dynamical problem of thermoelasticity for an anisotropic medium. *Arch. Mech. Stos.* **12** (1960), 137-147. (Polish and Russian summaries)

#### STRUCTURE OF MATTER

See also 6204, 6455, 6456.

6237:

Coldwell-Horsfall, Rosemary A.; Maradudin, Alexei A. Zero-point energy of an electron lattice. *J. Mathematical Phys.* **1** (1960), 395-404.

A calculation of the zero-point energy of a body centered cubic lattice of electrons in the harmonic approximation is made. Application is made of the moment-trace method as adapted to this type of calculation by Domb, Maradudin, Montroll, and Weiss [*Phys. Rev.* (2) **115** (1959), 18-36; MR **21** #6739; p. 24]. Using the first five non-vanishing moments the authors calculate a zero point energy of  $(2.638/r_s^{3/2})ry$  per electron and a low temperature specific heat per electron of  $56.21kr_s^{3/2}(kT)^{3/2}ry$ .

G. Weiss (Washington, D.C.)

6238:

Pippard, A. B. Theory of ultrasonic attenuation in metals and magneto-acoustic oscillations. *Proc. Roy. Soc. London. Ser. A* **257** (1960), 165-193.

Experiments on the passage of ultrasonic waves in metals at low temperatures have shown the existence of marked attenuation. In the presence of an external magnetic field the attenuation is dependent on the field strength. These effects are interpreted as arising from an interaction between the conduction electrons and the lattice when the latter is deformed by the passage of the wave. A mathematical theory is developed, based on the influence of the lattice deformation on the Fermi energy surface of the conduction electrons. The theory presumably is valid for frequencies below about  $10^9$  cycles/second, for which the deformation can be considered to occur quasi-statically. The results are complicated in detail, but show qualitative features in agreement with experiment, and with earlier theories based on the free-electron model of the conduction electrons.

E. L. Hill (Minneapolis, Minn.)

6239:

Turov, E. A.; Micek, A. I. Temperature dependence of magnetostriction. *Ž. Eksper. Teoret. Fiz.* **38** (1960),

1847-1851 (Russian. English summary); translated as Soviet Physics. *JETP* **11**, 1327-1330.

Authors' summary: "The temperature dependence of linear (anisotropic) and volume (isotropic) magnetostriction in the low-temperature region is studied on the basis of the phenomenological method of the theory of spin waves."

6240:

Boillet, Pierre. Extension du principe d'Huygens à un milieu discontinu: le solide considéré comme agrégat d'atomes. *C. R. Acad. Sci. Paris* **250** (1960), 3274-3276.

Author's summary: "It is shown that, in a solid formed of atoms, each having radius of action  $R$ , the vibrations propagated outside a closed surface containing a source are identical with those which would be obtained by suppressing the source and replacing it by suitable sources (secondary sources) situated on both sides of the surface at a distance from it less than  $R$ ."

E. T. Copson (St. Andrews)

6241:

Elbaum, C. On dislocations formed by the collapse of vacancy discs. *Phil. Mag.* (8) **5** (1960), 669-674.

Author's summary: "The formation of dislocations by the collapsing vacancy disc mechanism is examined for aluminium, copper, silicon and germanium. It is shown that this mechanism can be important in the case of metals. In the case of silicon and germanium dislocation loops produced by the above mechanism, during usual crystal growth from the melt, will not reach sizes detectable by means of the optical microscope."

W. R. Dean (London)

6242:

Kaplan, Jerome I. Correlation times, line widths, and cross relaxation of spin systems in solids. *Amer. J. Phys.* **28** (1960), 491-494.

Author's summary: "A simple computational formulation is derived for calculating line widths of electron (or nuclear) spin resonance absorption in solids. The method is not limited to a Zeeman local field energy. It is further shown how, starting with time-dependent perturbation theory, cross-relaxation times are calculated. In both cases the results are expressed as the Fourier transform of a correlation function."

6243:

Cohen, M. H.; Ham, F. S. Electron effective mass in solids—a generalization of Bardeen's formula. *Phys. and Chem. Solids* **16** (1960), 177-183.

Authors' summary: "A formula is derived for the effective mass of an electron in a crystal which replaces the sum over excited states in the usual sum rule by an integral over the surface of the unit cell. The integrand of the surface integral involves the wave function(s) at the symmetry point or band extremum  $k_0$  and a second solution of Schroedinger's equation at the same energy but satisfying inhomogeneous boundary conditions on the cell surface. The procedure is applicable regardless whether there is a degeneracy at  $k_0$ , and spin-orbit coupling may be taken into account. The result thus represents a generalization of Bardeen's formula for the

effective mass of an  $s$ -band at  $k=0$  in the Wigner-Seitz spherical approximation to an arbitrary band at any point  $k_0$ , using the polyhedral cell. A related variational principle for the components of the effective mass matrix is also derived."

6244:

Elliott, R. J. Some properties of concentrated and dilute Heisenberg magnets with general spin. *Phys. and Chem. Solids* **16** (1960), 165-168.

Author's summary: "The constant-coupling approximation is used to obtain the thermodynamic properties of a Heisenberg ferromagnet of general spin above the transition temperature. The variation of these properties with concentration of the magnetic atoms in an alloy are also given. Similar results for antiferromagnets are included in less detail with some critical discussion."

6245:

Van den Broek, J.; Gorter, C. J. The antiferromagnetic susceptibility at moderate fields. *Physica* **26** (1960), 638-646.

Authors' summary: "The isothermal differential susceptibility of an antiferromagnetic substance in a moderately strong magnetic field of arbitrary direction is calculated from the molecular field model as given by Gorter and Mrs. Van Peski-Tinbergen. The results are compared with the formulae of Nagamiya and Yosida and with experimental results on hydrated copper and manganese chlorides."

6246:

Keller, W. E.; Hammel, E. F., Jr. Heat conduction and fountain pressure in liquid He II. *Ann. Physics* **10** (1960), 202-231.

6247:

Godefroy, L.; Tavernier, J. Effets magnétoélectriques et thermomagnétoélectriques dans les semi-conducteurs. I. *J. Phys. Radium* **21** (1960), 249-260. (English summary)

Authors' summary: "In this paper, the electrical conductivity of a crystal in the presence of a magnetic field is investigated, by the method of the average energy gain. It has been possible to express the current density in tensor form, as a function of the electric and magnetic fields, in the case of a single valley. Also, in the limit of low magnetic fields, the resistivity and thermoelectric power tensors have been obtained (the latter includes the thermoelectric power proper, the Nernst and thermomagneto-resistance effects). The results are applied to the particular models of germanium and silicon."

H. Statz (Waltham, Mass.)

6248:

Greguss, Pál. On the relation between the ultrasonic velocity and the surface properties of metals. *Proc. Vibration Problems No. 5* (1960), 3-10. (Polish and Russian summaries)

6249:

Elliott, R. J.; Loudon, R. Theory of the absorption edge in semiconductors in a high magnetic field. *Phys. and Chem. Solids* **15** (1960), 196-207.

Author's summary: "A theory based on the effective mass approximation is given for hole-electron pairs in a semiconductor in a magnetic field. The intensity of the absorption edge is determined by the wave function of relative motion of the pair, so that essentially it becomes necessary to solve the Schrödinger equation for a hydrogen atom in a magnetic field. This is done in an approximation valid in high fields which assumes that the Coulomb term affects only the motion along the field, and uses a potential form which allows the solutions to be written in terms of confluent hypergeometric functions. The results show that the main intensity in each magnetic sub-band transition is thrown into the lowest exciton line and that the absorption in the continuum is reduced to an insignificant shoulder. The peaks observed in the so-called magneto-optic effect will all be exciton peaks."

6250:

Vaughan, Philip A. Regression formulae and the joint distribution of structure factors. *Acta Cryst.* **12** (1959), 981-987.

The author expresses the joint probability distribution of a set of structure factors by means of orthogonal polynomials. He is thus led to assume a particular form of regression formula for expressing a structure factor in terms of the magnitudes of other structure factors. By means of a least squares procedure he determines the constants in this expression and, after certain simplifying assumptions and approximations, finds a six-term expression in terms of observed magnitudes for the normalized structure factor  $E_H$  in space group  $Pl$  (where  $H = (h, k, l)$  and  $h, k$ , and  $l$  are all even). Although the six-term formula appears to be an improvement over some earlier results, comparison with more recent formulas is not made.

H. A. Hauptman (Washington, D.C.)

6251:

Ramachandran, G. N.; Raman, S. Syntheses for the deconvolution of the Patterson function. I. General principles. *Acta Cryst.* **12** (1959), 957-964.

The authors describe several Fourier syntheses which require not only a knowledge of the observed magnitudes,  $|F(H)|$ , of the structure factors (which are ordinarily obtainable experimentally), but also a partial knowledge of the crystal structure (which is often obtainable if the crystal contains a sub-structure consisting of a small number of heavy atoms). Denoting by  $F_p(H)$  the structure factors of the sub-structure (assumed known), they consider two main types of Fourier series in which the coefficients are  $|F(H)|^2 F_p(H)$  and  $|F(H)|^2 / F_p^*(H)$ , where  $F^*$  is the complex conjugate of  $F$ , as well as several auxiliary series. Except for unwanted background, these syntheses yield the crystal structure. (The extent to which the background interferes with the actual structure determination remains to be clarified by means of applications.)

In the case that the crystal consists of identical atoms, a least squares procedure for resolving the peaks in the Patterson function is described which requires only a

knowledge of the number of atoms in the structure. Owing to the enormous computations involved (except for the simplest structures) and the requirement of an accurate initial Patterson (which is not often obtainable) the practical utility of this procedure is somewhat dubious.

H. A. Hauptman (Washington, D.C.)

6252:

Raman, S. Syntheses for the deconvolution of the Patterson function. II. Detailed theory for non-centrosymmetric crystals. *Acta Cryst.* 12 (1959), 964-975.

The mathematical basis for the results described in the previous abstract is given.

H. A. Hauptman (Washington, D.C.)

#### FLUID MECHANICS, ACOUSTICS

See also A5774, 6194, 6195, 6216, 6365, 6394, 6542, 6597, 6598.

6253:

Quilghini, Demore. Generalizzazione di un teorema di Stokes sul potenziale gravitazionale di una massa fluida in equilibrio relativo e nuovo limite della velocità di rotazione. *Boll. Un. Mat. Ital.* (3) 14 (1959), 482-488. (English summary)

The theorem of Stokes for a figure of relative equilibrium  $\Sigma$  for incompressible fluids rotating rigidly with given angular velocity  $\omega$  about a given axis states that exterior to  $\Sigma$  all such potentials are the same, whatever the density within  $\Sigma$ . The author generalizes this result by relaxing the requirement of relative equilibrium. He proves that the same conclusion follows when one potential is assumed to correspond to equilibrium, while the second is that due to the same total mass distributed homogeneously within  $\Sigma$ . The proof, which is not easy, rests on the theorem of Lichtenstein. The author points out that when Crudeli's argument is applied to this result, the celebrated theorem of Crudeli is then automatically generalized to inhomogeneous figures in the following form:  $\omega^2 < \pi f \rho^*$ , where  $f$  is the gravitational constant and  $\rho^*$  is the mean density. This is notably stronger than Crudeli's own result for heterogeneous figures, namely,  $\omega^2 < \pi f \rho_{\max}$ .

C. Truesdell (Basel)

6254:

Gheorghitza, St. I. A semi-inverse method in plane hydrodynamics. *Arch. Mech. Stos.* 11 (1959), 681-689. (Polish and Russian summaries)

Consider a plane body which occupies the domain  $D$  interior to a contour  $C$ . Let the region exterior to  $D$  have a known mapping on a half-plane. If we replace an arc  $AB$  of  $C$  by a curve  $APB$ , the mapping will produce a boss on the boundary of the half-plane. If we assign the shape of this boss so that the resulting diagram can be mapped on a half-plane, the form of the curve  $APB$  can be inferred. It is then possible to determine the perturbation of given inviscid irrotational flow of liquid due to the body  $D$  modified by the arc  $APB$ .

L. M. Milne-Thomson (Madison, Wis.)

6255:

Lighthill, M. J. Note on the swimming of slender fish. *J. Fluid Mech.* 9 (1960), 305-317.

The author's summary reads: "The paper seeks to determine what transverse oscillatory movements a slender fish can make which will give it a high Froude propulsive efficiency,

$$\frac{(\text{forward velocity}) \times (\text{thrust available to overcome frictional drag})}{(\text{work done to produce both thrust and vortex wake})}$$

The recommended procedure is for the fish to pass a wave down its body at a speed of around  $5/4$  of the desired swimming speed, the amplitude increasing from zero over the front portion to a maximum at the tail, whose span should exceed a certain critical value, and the waveform including both a positive and a negative phase so that angular recoil is minimized. The Appendix gives a review of slender-body theory for deformable bodies." He also considers briefly the possible effects of the boundary layer on his results.

J. W. Miles (Los Angeles, Calif.)

6256:

Milne, R. D. The aerodynamic forces associated with harmonic motion of slender wings on stationary bodies of revolution. *Aero. Quart.* 11 (1960), 355-370.

Author's summary: "The slender-body theory is applied to the determination of the unsteady aerodynamic forces on slender wing-body combinations when the body is stationary and the wing is deforming harmonically. The uniform body is circular in cross section, the wings are placed symmetrically and the wing deformation mode is necessarily symmetrical about a vertical plane through the body axis. The frequency parameter is restricted to that range for which the 'cross-flow' potential equation is Laplace's equation. The case of an all-moving slender control surface on a body is treated in detail and numerical results are given for the forces and moments on body and control surface when the control plan form is triangular; the presence of a gap is neglected."

6257:

Rodden, William P.; Revell, James D. Oscillatory aerodynamic coefficients for a unified supersonic-hypersonic strip theory. *J. Aero/Space Sci.* 27 (1960), 451-459.

The reviewer's second-order (in thickness) theory for oscillating airfoils in supersonic flow is generalized to swept wings with supersonic leading edges, and the aerodynamic coefficients calculated for a streamwise strip. Agreement with other theories is shown in limiting cases.

M. D. Van Dyke (Stanford, Calif.)

6258:

Kovaleva, V. A. On the nonstationary motion of a wing with rectangular planform. *Prikl. Mat. Meh.* 23 (1959), 1030-1041 (Russian); translated as *J. Appl. Math. Mech.* 23, 1476-1491.

The disturbance potential for unsteady motion of a thin, lifting, rectangular wing at supersonic speed is determined, first for the case when the downwash at the wing surface is  $\partial\phi/\partial z = e^{\alpha t}$ ,  $\alpha$  being a constant,  $t$  the time. This is accomplished by use of results obtained by A. Busemann [*Schr. Deutsch. Akad. Luftfahrtforschung* 7B (1943), 105-121; MR 8, 415] and L. A. Galin [*Prikl. Mat. Meh.* 11 (1947), 465-474; MR 9, 254], and Laplace-transform techniques. From the potential the pressure

and lift are calculated. Next, the case of a wing of infinite span in sinusoidal motion is worked out and it is verified that the results are obtained as a limiting case of what has been described above. Finally, the finite-span result is used to determine the flow due to a unit-step vertical gust. Here the perturbation potential is calculated in the form of an infinite series in powers of  $\epsilon^2$ .

W. R. Sears (Ithaca, N.Y.)

6259:

Lisunov, A. D. On the theory of unsteady supersonic gas flow about a wing of finite span. Prikl. Mat. Meh. 24 (1960), 166-168 (Russian); translated as J. Appl. Math. Mech. 24, 230-232.

For certain classes of thin, lifting, supersonic wings the perturbation potential at a point on the surface is given by a surface integral involving a source distribution  $q(x, y, t)$  in which the time variable  $t$  is replaced by appropriate values representing the effects of acoustic propagation. Here this time correction is expressed by use of an exponential differential operator; i.e.,  $f(t \pm \tau) = \exp(\pm \tau \partial/\partial t)f(t)$ . Upon writing this operator formally as a series expansion in both of its appearances within the double integral, the author obtains the perturbation potential in the form of a series with respect to the orders of time derivatives. The functions differentiated are surface integrals involving  $q(x, y, t)$ . The first term (involving no differentiation) gives steady-flow theory; retention of two terms yields quasi-steady theory.

It is shown here, but very concisely, how this result can be used to determine the response of an elastic wing to a given vertical-gust velocity  $W(x, y, t)$ . In practice the series is to be terminated at a finite number of terms "depending on the capabilities of the electronic computer", and a number of functions of time, which are the deflections in various modes, are to be found.

W. R. Sears (Ithaca, N.Y.)

6260:

Forster, C. A.; Southgate, A. C. The difference function approach to the overall aerodynamics of guided missiles. J. Roy. Aero. Soc. 64 (1960), 753-763.

6261:

Yih, Chia-Shun. Exact solutions for steady two-dimensional flow of a stratified fluid. J. Fluid Mech. 9 (1960), 161-174.

The equations for steady motion of a stratified fluid under gravity in vertical planes (2 dimensions) are linear even when the amplitude is large, if the velocity and density profiles of the undisturbed stream have certain shapes. Periodic waves are therefore possible, and closed stationary circulations can occur with the waves flowing round them. The variety of profiles for which the equation is linear is considerably extended beyond those already discovered by Long. The fact (already well known to meteorologists) that the amplitude of the lee waves does not depend on the precise form of an obstacle shape, but on an integral property of it, is neatly expressed.

Although particular, the profiles are quite varied, and can be used to study many different flows in stratified liquids with free surface or rigid lid. Applications to the atmosphere present additional difficulties because the profile over the whole infinite depth must have a single

formulation. However, these solutions give wider validity to the perturbation methods hitherto applied to the atmosphere.

R. S. Scorer (London)

6262:

Yih, Chia-Shun. Gravity waves in a stratified fluid. J. Fluid Mech. 8 (1960), 481-508.

This very interesting paper covers many aspects of its subject, not all of which can be mentioned in this review. The equations for a compressible medium are derived and specialized to plane waves. It is shown that if the velocity of sound is taken to be constant, the equations for compressible and incompressible media are similar. Horizontal plane wave motion in an incompressible medium is then treated when the fluid is bounded above and below by rigid horizontal boundaries. The stream function then satisfies a Sturm-Liouville system, modified if there are finite discontinuities in density. Sturm's oscillation theorem is applied to give (among other results) upper and lower bounds for the phase velocity for a given stratification, a given wave number, and a given number of zeros of the eigenfunction, which are used to explain the well-known tendency for internal surfaces of density discontinuity to behave as rigid boundaries when the stratification in each layer is slight. The spectrum can be obtained approximately from this kind of consideration, while more precise results are obtainable from the Rayleigh-Ritz method, modified to take account of discontinuities. The nature of the spectrum for infinite depth is also treated. Other topics include the wave motion due to a plane vertical wavemaker, and the stability of stratified liquid under vertical vibration.

F. Ursell (Cambridge, England)

6263:

Späenberg, J. A. The influence of surface tension on the surface waves induced by a rolling thin strip. Nederl. Akad. Wetensch. Proc. Ser. B 63 (1960), 335-352.

The ordinary gravity waves produced on an infinite free water surface by the rolling of a rigid strip on it radiate energy. It is concluded on the other hand, in this study, that the surface tension waves tend to zero at infinite distance from this strip, but the physical meaning of this conclusion is not elucidated, and the effect of surface tension on model experiments of a rolling strip not made plain, although this is the author's stated objective.

R. S. Scorer (London)

6264:

Ursell, F. Steady wave patterns on a non-uniform steady fluid flow. J. Fluid Mech. 9 (1960), 333-346.

The wave pattern produced by a point disturbance on a free surface ('ship waves') can be derived using only kinematic considerations together with a knowledge of how the phase speed depends on wave number. In this paper the derivation for a fixed point on a steady stream is based on two main assumptions: (1) the stream velocity component normal to a wave crest is equal to the phase velocity based on local wave number (this is the condition that the wave crest is steady); (2) the separation between consecutive crests is equal to the local wave length (a more direct statement is that the vector wave number is irrotational, because it is the gradient of the

phase function). The first of these gives a functional relation for the two components ( $k_1, k_2$ ) of vector wave number; (2) gives  $\partial k_2/\partial x_1 - \partial k_1/\partial x_2 = 0$ . Thus ( $k_1, k_2$ ), and therefore the whole wave pattern, can be determined.

This theory allows wave patterns on non-uniform streams to be determined, and it is applied in particular to the wave pattern on a thin sheet of fluid expanding with constant radial velocity and thickness varying inversely with distance from the origin. This example arises in Taylor's study of waves on fluid sheets [Proc. Roy. Soc. London. Ser. A **253** (1959), 289-295; MR **22** #1242].  
G. B. Whitham (Cambridge, Mass.)

6265:

Peržnyanko, È. A. The problem of the wave resistance of a body moving in a circle. Dokl. Akad. Nauk SSSR **130** (1960), 514-516 (Russian); translated as Soviet Physics. Dokl. **5**, 43-45.

A prescribed rigid system of moving sources is moving horizontally in a circular path with constant angular velocity under the free surface of an ideal heavy liquid. The wave motions which it generates in a circular basin, in a circular channel and outside a circular cylinder are treated according to the linearized theory of surface waves; it is assumed that the motion has become steady relative to the sources. In this brief note certain results are stated but no derivation is given.

F. Ursell (Cambridge, England)

6266:

Phillips, O. M. On the dynamics of unsteady gravity waves of finite amplitude. I. The elementary interactions. J. Fluid Mech. **9** (1960), 193-217.

The author considers the second- and third-order interactions between two intersecting sinusoidal wave trains of wave numbers  $k_0$  and  $k_1$  on the surface of deep water. He finds that the secondary components are bounded in time but that certain tertiary components of wave number  $2k_0 - k_1$  may grow linearly in time, extracting energy from the primary waves. He also finds that the amplitude of the secondary components may be large, albeit still bounded in time, in shallow water. The analysis is carried out using Fourier-Stieltjes transforms in anticipation of subsequent generalization to a random sea.

J. W. Miles (Los Angeles, Calif.)

6267:

Larras, J.; Chapon, J. Conditions générales d'amarrage des navires avec application particulière au cas des seiches et du mascaret. Ann. Ponts Chaussées **130** (1960), 747-766. (English summary)

Authors' summary: "The first part of this article contains a mathematical study of the general conditions of mooring a ship in a dock where the water oscillates in the same direction as its length-wise axis.

"The authors then extend their study to deal with seiches and tidal waves or bores in river estuaries. This brings out the fundamental part played by the period  $\omega_0 = 2\pi/\omega_0$  of the movement of the ship on its hawsers quite apart from any movement of the water, concurrently with that of the period  $\theta = 2\pi/\omega$  of the beginning of the thrust of the water against the ship. In particular, care must be taken that the differences  $\omega_0^2 - \omega^2$  and  $\omega_0^2 - 4\omega^2$  deviate only slightly from  $\omega_0^2$  when it is desired to reduce

to the greatest possible extent the ship's to and fro movement and the drag on the hawsers.

"The authors then compare the requirements resulting from their calculations with the findings by actual measurements on a 10,000 ton tanker in a bore in the Seine estuary. This comparison gives experimental confirmation of the requirements found by calculation and, among the latter, the importance of a system of hawsers of great elasticity but slightly pre-tensioned to ensure the ship will stand up to the bore under the best possible conditions."

6268:

Sinha, C. P. Flow near the point of separation. Proc. 4th Congress Theoret. Appl. Mech. 1958, pp. 145-150. Indian Soc. Theoret. Appl. Mech., Kharagpur.

Author's summary: "Two dimensional viscous fluid flow is discussed for the particular case in which the stream-function is of the form  $\psi = f(y) + xg(y)$ . The conditions for the separation of the flow at the leading edge, taken at the origin, are also discussed."

6269:

Ghildyal, C. D. Unsteady motion of a viscous liquid contained between two infinite coaxial cylinders  $r=a$  and  $r=b$ . Z. Angew. Math. Mech. **39** (1959), 473-476. (German, French and Russian summaries)

When  $t < 0$  the motion is steady, the outer cylinder being rotated with constant angular velocity and the inner being at rest. When  $t \geq 0$  the inner cylinder is also rotated with constant angular velocity; the velocity of the liquid, which in transverse throughout, is expressed for all  $r, t$  as an infinite series in terms of Bessel functions.

W. R. Dean (London)

6270:

Sparrow, E. M.; Gregg, J. L. Flow about an unsteadily rotating disc. J. Aero/Space Sci. **27** (1960), 252-256, 290.

The subject of analysis is the laminar flow of a viscous incompressible fluid bounded by an infinite disc which rotates in its plane at a varying angular velocity  $\omega$ . On the assumption that  $\omega$  varies only gradually, a solution is sought in the form of a perturbation from the corresponding sequence of steady states; accordingly, the velocity components and pressure are expanded in terms of  $\dot{\omega}/\omega^2, \ddot{\omega}/\omega^3, \dots$  which are taken to be of successively diminishing magnitude. The authors have calculated the steady-state solution (due originally to von Kármán) with greater accuracy than previously attained, and they also give numerical data for the higher-order approximations. As the principal result of the analysis, the leading terms in the expansion of the shear stress on the disc are presented. With regard to calculations for practical purposes, it is shown that if  $\dot{\omega} \leq 0.0834\omega^2$  the shear stress can be estimated to within 5 per cent by considering the flow to be quasi-steady.

T. B. Benjamin (Cambridge, England)

6271:

Gorelik, L. V. On the connection between viscous friction forces and flow potential. Ž. Tehn. Fiz. **30** (1960), 653-655 (Russian); translated as Soviet Physics. Tech. Phys. **5**, 615-617.

Author's summary: "The dependence of the viscous friction force acting on a fluid flowing through a porous body on the gradient of the electric potential arising as a consequence of the appearance of a flow potential in laminar flow of a polar fluid through a porous body is obtained."

6272:

Payne, L. E.; Pell, W. H. The Stokes flow problem for a class of axially symmetric bodies. *J. Fluid Mech.* 7 (1960), 529-549.

A general treatment is given of the slow steady axisymmetrical motion of incompressible viscous liquids. The stream function is found for the flow past a general lens-shaped body, and considered in more detail for a body in the form of a hemisphere, of a symmetrical biconvex lens and of a spherical cap; in the last case a simple expression is found for the force acting on the body.

W. R. Dean (London)

6273:

Cooke, J. C. Boundary layers over infinite yawed wings. *Aero. Quart.* 11 (1960), 333-347.

Author's summary: "A method of calculating turbulent boundary layers on infinite yawed wings is given, making use of a method of calculating turbulent boundary layers due to Spence and of an analogy between three-dimensional and axisymmetric boundary layers. It is also shown that the displacement thickness is equal to that computed using chordwise components and that the streamwise momentum thickness is approximately equal to the chordwise momentum thickness. Shock-free flow and small boundary layer cross-flow are assumed. Good agreement is shown with the observed growth of momentum and displacement thickness in the boundary layer on a particular wing at 55° sweepback."

D. A. Spence (Farnborough)

6274:

Hayasi, Nisiki. On similar solutions of the steady quasi-two-dimensional incompressible laminar boundary-layer equations. *J. Phys. Soc. Japan* 15 (1960), 522-527.

Boundary layer equations in orthogonal curvilinear coordinates are used for investigating flows in which one component of the velocity and the pressure gradient in that direction are negligible throughout the boundary layer.

New similar solutions are given for flows along curved surfaces, on which an orthogonal system of curvilinear coordinates  $\alpha, \beta$  is defined and for an exterior flow in which the velocity at the outer edge of the boundary layer can be expressed as  $(B_1 G(\alpha, \beta) + B_2)^m$  or  $B_2 \exp\{B_1 G(\alpha, \beta)\}$ .  $B_1$  and  $B_2$  are constants, and  $G(\alpha, \beta)$  is given by the surface geometry.

The results are used to study some particular cases, for instance the boundary layer flow connected with a three-dimensional inviscid flow in a half space bounded by a flat plate and having radial streamlines on the plate. An extension to other than plane boundaries is discussed.

I. Flügge-Lotz (Stanford, Calif.)

6275:

Spence, D. A. A note on the recovery and Reynolds-analogy factors in laminar flat-plate flow. *J. Aerospace Sci.* 27 (1960), 878-879.

Analytical approximations for the recovery factor  $r$  and Reynolds-analogy factor  $s = 2c_h/c_f$  are derived for values of the Prandtl number  $\sigma$  close to one. The viscosity is assumed proportional to the absolute temperature, so that a non-dimensional normal distance  $\eta$  and a non-dimensional stream function  $f(\eta)$  can be defined, where  $f(\eta)$  is the usual Blasius solution for the incompressible boundary layer. The quantities  $r$  and  $s$  can then be expressed as definite integrals involving  $f(\eta)$  and the parameter  $\sigma$ . For  $\sigma$  close to one, these lead to the approximate relations  $r = \sigma^{1/2}[1 + 0.0095(\sigma - 1)^2 + \dots]$  and  $s = \sigma^{-n}[1 - 0.347(\sigma - 1)^2 + \dots]$ , where  $n = 0.64885$ .

D. W. Dunn (Ottawa, Ont.)

6276:

Spence, D. A. Velocity and enthalpy distributions in the compressible turbulent boundary layer on a flat plate. *J. Fluid Mech.* 8 (1960), 368-387.

The author derives the mean-flow properties starting from two empirical laws for the velocity profile. The outer part of the profile is described by  $u/u_\infty = (\eta/\Delta)^{1/n}$ , where  $\eta = \int_0^\eta (\rho/\rho_0) dy$  is the Howarth variable and  $\Delta$  the corresponding boundary-layer thickness. The inner part fits a "law of the wall",  $u/u_\tau = A \log(\eta u_\tau/\nu_0 - c) + B$ , where  $u_\tau = (\tau_w/\rho_0)^{1/2}$ ,  $\rho_0$  and  $\nu_0$  are evaluated at a suitable "intermediate" enthalpy, and  $A, B$ , and  $c$  are the same constants as in incompressible flow. With the use of these relations, the remaining boundary-layer characteristics are determined analytically from the equation of motion and the energy equation. The result for the shear-stress distribution is approximately  $\tau/\tau_w = 1 - (u/u_\infty)^{n+2}$ . The expression for the enthalpy is similar to Crocco's integral for the laminar boundary layer, with a turbulent Prandtl number  $\alpha$  (assumed constant) in place of the laminar Prandtl number  $\sigma$ , but contains two extra terms proportional respectively to  $(\alpha - \sigma)c_f$  and  $(\alpha - \sigma)c_f^{1/2}$ . The corresponding heat-transfer coefficient in terms of recovery factor and skin-friction coefficient agrees well with available experimental data. The usual quadratic enthalpy-velocity relation, exact for  $\alpha = \sigma = 1$ , remains an acceptable approximation for Prandtl numbers considerably different from unity.

D. W. Dunn (Ottawa, Ont.)

6277:

Mežirov, I. I. On the turbulent boundary layer of an imperfect gas. *Prikl. Mat. Meh.* 24 (1960), 93-99 (Russian); translated as *J. Appl. Math. Mech.* 24, 120-128.

The equations of momentum, energy and state for turbulent flow in a boundary layer containing a mixture of species—for example, dissociated air—are set out in a neat form from which density fluctuations have been removed by defining, for a fluctuating variable  $q$ , the mean value  $q^0 = \overline{\rho q}/\bar{\rho}$ . These are of the same form as those for a perfect gas, but some further knowledge of the diffusion processes for the different species—at present lacking—is necessary before the average molecular weight, which enters the equation of state, can be known. The connection between velocity distribution and heat transfer is established by integrating the energy equation assuming the "turbulent Prandtl number", i.e., the ratio of the eddy diffusivities of momentum and heat, to be constant across the layer. The results are similar to those obtained by the reviewer in a recent note [see preceding review] using the same assumption. Physically it seems

unlikely that the detailed mechanisms of heat and momentum transport will be so similar throughout the boundary layer as to keep the ratio of diffusivities constant, but the assumed value of  $(Pr)_{\text{turb}}$  may nevertheless be looked on as a weighted mean from which the deviations should not be too large. *D. A. Spence (Pasadena, Calif.)*

6278:

**Miles, John W.** The hydrodynamic stability of a thin film of liquid in uniform shearing motion. *J. Fluid Mech.* 8 (1960), 593-610.

The stability problem for a thin film of liquid having linear velocity profile with a free upper surface is solved asymptotically for large values of the Reynolds number. Such a configuration might arise in film cooling and nose-cone ablation. The dynamic effects of the lighter upper fluid on the thin layer are also discussed.

The analysis of the stability problem is similar to that for classical plane Couette flow; however, instability is found in this case for sufficiently large Reynolds numbers. Neutral stability curves for Reynolds number vs. wave number are given for several values of the Weber number (based on mean speed at the surface and the depth of the layer).

In an appendix results for the Tietjens function are given. *R. C. DiPrima (Troy, N.Y.)*

6279:

**Sorokin, V. S.; Suškin, I. V.** Stability of equilibrium of a conducting liquid heated from below in a magnetic field. *Ž. Èksper. Teoret. Fiz.* 38 (1960), 612-620 (Russian. English summary); translated as Soviet Physics. *JETP* 11, 440-445.

From authors' summary: "The effect of a uniform magnetic field on the stability of the equilibrium of a conducting liquid heated from below in a cavity of arbitrary shape is investigated. The critical value of the Rayleigh number  $C_0^2$ , above which the equilibrium is unstable, increases monotonically with the Hartmann number  $M$ , and at small values of  $M$  is proportional to  $M^2$ . The asymptotic nature of the function  $C_0(M)$  as  $M \rightarrow \infty$  depends on the shape of the cavity and the direction of the field." *D. W. Dunn (Ottawa, Ont.)*

6280:

**Rumyantsev, V. V.** The stability of the rotational motions of a solid body with a liquid cavity. *Prikl. Mat. Meh.* 23 (1959), 1057-1065 (Russian); translated as *J. Appl. Math. Mech.* 23, 1512-1524.

The author considers the stability of rotational motion of a solid body with a cavity filled entirely, or partially, with an ideal incompressible homogeneous liquid. The type of stability established here is temporary stability, i.e., stability which can be achieved with gyroscopic stabilization. *H. P. Thielman (Oxnard, Calif.)*

6281:

**Phillips, O. M.** Centrifugal waves. *J. Fluid Mech.* 7 (1960), 340-352.

If a circular cylindrical vessel closed at both ends and partially filled with water spins about its axis, the water

would, in the absence of gravity, eventually rotate like a rigid body about a central air core. The paper analyses two forms of perturbation from the exact rigid-body rotation: first, the steady perturbation due to gravity when the axis is horizontal; second, the modes of oscillation (centrifugal waves) of the free surface. With regard to the first effect a criterion of stability is derived, and this is checked experimentally. Although the theory is confirmed over a certain range of conditions, the experiments showed that instability and collapse of the approximately rigid-body rotation sometimes occurred prematurely, following the spontaneous formation of large amplitude centrifugal waves. The theory does not account for the excitation of these waves (though presumably it occurs by coupling between the two forms of perturbation which are analysed separately); but some measurements of wave frequencies are shown to be in excellent agreement with the theoretical values. Credit is due to the author for having initiated a problem of considerable theoretical and practical interest which well deserves further study.

*T. B. Benjamin (Cambridge, England)*

6282:

**Bryan, Kirk.** The instability of a two-layered system enclosed between horizontal, coaxially rotating plates. *J. Meteorol.* 17 (1960), 446-455.

Two immiscible fluids of slightly different densities are placed in an open cylinder and rotated about a vertical axis. At the same time a glass plate acting as a lid to the cylinder is set rotating about the same axis but with a slightly different angular velocity. The author calculates the angular velocity distribution in the fluids using the theory of Ekman boundary layers. He then examines the stability of the interface using inviscid arguments. A comparison of his conclusions with experiment gives quite good agreement. Finally, the relevance of the work to the theory of baroclinic instability in meteorology is discussed.

*K. Stewartson (Durham)*

6283:

**Batchelor, G. K.** ★The theory of homogeneous turbulence. Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge University Press, New York, 1959. xi+197 pp. Paperbound: \$3.75.

Paperbound reprinting as "Students' edition" of the first edition [1953; MR 14, 597].

6284:

**Rosen, Gerald.** Turbulence theory and functional integration. I, II. *Phys. Fluids* 3 (1960), 519-524, 525-528.

Hopf's equation of motion for the characteristic functional of the velocity distribution in incompressible turbulence [*J. Rational Mech. Anal.* 1 (1952), 87-123; MR 15, 478] is integrated to give formal expressions, for moments, in the form of functional integrals. As presented, the integrals appear to involve singularities which make them mathematically ill-defined. The reviewer is unable to justify an approximate evaluation of the velocity covariance, given for the case of a singular initial velocity distribution in which  $E(k) \propto k^3$  for all  $k$ , where  $E(k)$  is the usual wave-number spectrum of kinetic energy.

*R. H. Kraichnan (New York)*

6285:

Saffman, P. G. On the effect of the molecular diffusivity in turbulent diffusion. *J. Fluid Mech.* 8 (1960), 273-283.

The Lagrangian correlation between the velocity of a fluid particle at different times is generalized to describe the dispersion of a substance with appreciable molecular diffusivity  $k$  by defining a "substance auto-correlation function"

$$S_d(\tau) = \overline{V(t)V(t+\tau)}/\overline{V^2},$$

where  $V(t)$ ,  $V(t+\tau)$  are the fluid velocities at times  $t$  and  $t+\tau$  at the positions occupied by a single molecule at these times. The observed dispersion is then

$$D^2 = 2\overline{V^2} \int_0^{t-t_0} (t-t_0-\tau) S(\tau) d\tau + 2k(t-t_0)$$

and interaction between the molecular and turbulent diffusions depends on the difference between  $S_d$  and the ordinary Lagrangian correlation,  $S_p(\tau) = \overline{v(t)v(t+\tau)}/\overline{v^2}$ . The difference  $S_p - S_d$  is calculated for small intervals of order  $\tau^2$ , and it is shown that the effect of interaction is to reduce dispersion by  $k\omega^2(t-t_0)^2/q$ , where  $\omega^2$  is the mean square vorticity. The behaviour for large values of  $\tau$  is also considered by more qualitative arguments and the reduction in dispersion is found to be proportional to  $kR_\lambda^{-1}/v$ . This is in agreement with measurements by Mickelsen of the dispersion of carbon dioxide and helium in a turbulent air-flow.

A. A. Townsend (Cambridge, England)

6286:

Chou, Pei-yuan. Similarity structure of vorticity fluctuation and the theory of turbulence. *Sci. Sinica* 8 (1959), 1095-1119.

Since the scale of vorticity fluctuations in a turbulent flow is much less than the scale of the mean velocity distribution, it is possible to separate the equations for the mean motion and for the turbulent vorticity fluctuations. The author writes down the equations for the vorticity fluctuation in a coordinate system moving with the local mean velocity and substitutes in them the similarity solution,  $w_i' = q\phi_i(\xi)$ ,  $\xi_i = x_i'/\Lambda$ ,  $q^2 = w_j'w_j'$ , where  $q$  is the root-mean-square of the velocity fluctuation and  $\Lambda$  is the scale of the vorticity. Substituting, he obtains an equation in the non-dimensional variables,  $\phi_i$  and  $\xi_i$ , and similarity solutions exist if the coefficients are in constant ratio. Results, which depend on the possibility of neglecting various terms, are obtained for flow in pipes, channels, boundary-layers and wakes, giving the von Kármán similarity law for wall flow and constant eddy-viscosity for wakes. The agreement with measurement is satisfactory for the mean flow but less so for the turbulent intensities. The method is also applied to temperature distributions. A. A. Townsend (Cambridge, England)

6287:

Tanenbaum, B. Samuel; Mintzer, David. Energy transfer in a turbulent fluid. *Phys. Fluids* 3 (1960), 529-538.

Von Kármán has suggested that the transfer of turbulent energy from one Fourier component to another may be described by the transfer function,

$$T(k) = 2\gamma[k^m E^n \int_0^k k'^{1/2-m} E^{3/2-n}(k') dk' - k^{1/2-m} E^{3/2-n} \int_k^\infty k'^m E^n(k') dk'],$$

where  $E(k)$  is the energy spectrum function.  $T(k)$  is the net transfer of energy from components of wave-number  $k$ . The validity of this suggestion has been assessed by computing the related function,  $S(k) = \int_{-\infty}^\infty T(k') dk'$ , from measurements of the triple correlation function in grid turbulence and comparing it with values computed from the measured spectrum functions using the von Kármán equation for three choices of  $m, n$ . Best agreement is found for  $m = -5/2$ ,  $n = 1/4$  for measurements behind a grid at a Reynolds number of 5300 and values of  $x/M$  from 20 to 120. These values imply that transfer of energy in wave-number space is more localized than the eddy viscosity hypothesis of Heisenberg indicates.

A. A. Townsend (Cambridge, England)

6288:

Howells, I. D. An approximate equation for the spectrum of a conserved scalar quantity in a turbulent fluid. *J. Fluid Mech.* 9 (1960), 104-106.

In a turbulent flow with a small Prandtl number, the spectrum of temperature fluctuations has two limiting forms in the equilibrium range, depending on whether the wave-number is less than or greater than  $(\epsilon/k^3)^{1/4}$  ( $\epsilon$  is energy dissipation,  $k$  thermometric conductivity). Both limiting forms can be described by a modified form of the Heisenberg "eddy viscosity" form for the spectral transfer-function,

$$k_\epsilon(n) = [k^2 + \frac{4}{3} \int_n^\infty E(n)/n^2 dn]^{1/2} - k,$$

where  $E(n)$  is the velocity spectrum. Reasons are given for expecting wide applicability of this form for the eddy transfer coefficient, but it is pointed out that the concept is not applicable to the velocity spectrum itself.

A. A. Townsend (Cambridge, England)

6289:

v. Krzywoblocki, M. Z. ★Bergman's linear integral operator method in the theory of compressible fluid flow. With an appendix by Ph. Davis and Ph. Rabinowitz. Springer-Verlag, Vienna, 1960. x+188 pp. \$12.40.

Let  $L$  denote a linear partial differential operator and  $L_1$  a related but more complicated operator. In a long series of papers Stefan Bergman has expounded methods for finding solutions of  $L_1[\psi] = 0$  in the form of integrals, whose integrands involve an arbitrary solution of  $L[\psi] = 0$ ; this is the so-called 'integral operator' method. For application to the two dimensional flow of an inviscid compressible fluid the independent variables may be so chosen that  $L$  is the Laplacian, and the procedure generates (within its region of convergence) a 'compressible' stream function from an arbitrary 'incompressible' one. The author of the book under review notes (p. 173) that Bergman's method has largely been neglected by aerodynamicists, and states in his preface that his object is 'to represent the method in all its variations in such a way that a theoretical engineer or an applied aerodynamicist can use it in practical applications'. To this end 'the emphasis was put upon the simplified presentation of the final results and formulas, rather than upon the derivation of these formulas'.

The author hardly attempts a critical or comparative assessment of Bergman's work. The book in fact consists largely of extracts, almost verbatim, from Bergman's and related papers. They are strung together on a sensible

plan, but the omissions and alterations sometimes lead to obscurities, and they are sometimes provoking, e.g., when figures are omitted or the paragraphing of the original is altered. From the resulting macedoine the attentive reader may form a fair idea of the character, scope and actual or potential achievement of Bergman's work, but for a critical comparison, in these matters, with other methods he will have to rely very largely on his own knowledge of the latter.

The reviewer has found the easiest reading in Parts I and II (which deal with subsonic flow and are largely a reprint from v. Mises and Schiffer's article in *Advances in applied mechanics*, vol. 1, pp. 249-285 [Academic Press, New York, 1948; MR 10, 642]), and Part XV by Davis and Rabinowitz (on numerical computations). At the other extreme the author's method—and perhaps also Bergman's—seems to be least successful in Part IV, on transonic flow.

T. M. Cherry (Melbourne)

6290:

Pavlov, K. B. On the theory of motions of the Prandtl-Meyer type. *Prikl. Mat. Meh.* 24 (1960), 165-166 (Russian); translated as *J. Appl. Math. Mech.* 24, 227-229.

Suppose that the stream flowing toward an edge is not quite uniform. The flow pattern can then be calculated as a slightly perturbed Prandtl-Meyer flow. The perturbation quantities are given by three simultaneous, linear, first-order differential equations in two independent variables (plane polar coordinates). By matrix methods these are reduced to three linear, first-order equations such that the three dependent variables can be determined successively. No examples are presented, but the equations are written out in detail.

W. R. Sears (Ithaca, N.Y.)

6291:

Gundersen, Roy. The non-isentropic perturbation of an arbitrary simple wave. *J. Math. Mech.* 9 (1960), 141-145.

The quasilinear system for  $u, c, s$  (particle and sound velocity, entropy) in independent variables  $x$  and  $t$ , describing the non-homentropic one-dimensional flow of a perfect gas, is linearized with respect to a perturbation  $u_1, c_1, s_1$  of a homentropic solution  $u_0, c_0$  with Riemann invariants  $\alpha, \beta$ . Along the particle lines of the basic flow  $s_1$  is then constant and may be prescribed; the linear combinations  $\pm u_1/2 + c_1/(\gamma - 1)$ , abbreviated  $A$  and  $B$ , satisfy a pair of inhomogeneous linear first-order equations, in which  $A$  as well as  $B$  are multiplied by  $\alpha_x/\beta_x$ , in the first [second] equation and only the derivatives of  $A$  or  $B$  occur in each equation. The right-hand sides depend on the entropy distribution across the particle lines. Thus, if the basic flow is a simple wave, the equations are uncoupled. In this case the general solution can be written in terms of quadratures and three arbitrary functions, one of which is the entropy distribution across the particle lines.

G. Kuerti (Cleveland, Ohio)

6292:

Fox, R. H. An accurate expression for gas-pressure drop in high-speed subsonic flow with friction and heating. *J. Appl. Mech.* 27 (1960), 747-748.

1666

6293:

Bagdov, A. G. New method of determining the pressure in a fluid. *Akad. Nauk Armyan. SSR. Dokl.* 29 (1959), 153-157. (Russian. Armenian summary)

The propagation of the pressure-wave in a non-homogeneous compressible fluid occupying a half space is considered, when the impulse is going out from the point  $P$  on the boundary. If the  $r$ -axis is supposed to be in the boundary plane and the  $z$ -axis perpendicular to it, the pressure  $P$  in an arbitrary point of the fluid will be determined by means of the wave-equation

$$(1) \quad \frac{\partial^2 P}{\partial r^2} + \frac{\partial^2 P}{\partial z^2} + \frac{1}{r} \frac{\partial P}{\partial r} = \frac{1}{a^2(z)} \frac{\partial^2 P}{\partial t^2},$$

with given boundary conditions. It is assumed that the function  $1/a^2(z)$  can be developed in a power-series of  $kz$ , where  $k$  is a parameter characterizing the non-homogeneity of the fluid.

The solution of the equation (1) is sought in the form  $P = P_0 + kP_1 + k^2P_2 + \dots$ . When this is put into (1) and the coefficients of equal powers of  $k$  equated, one obtains the differential equations determining  $P_0, P_1, \dots$ . The evaluation is then performed by introducing the Laplace transforms. The author claims that his method gives better results than the classical separation of variables in the equation (1) or the methods related to Hadamard's method [*Le problème de Cauchy*, Hermann, Paris, 1932]. It is, however, not clear for what reason there is no mention of the convergence of the obtained series.

T. P. Andelić (Belgrade)

6294:

Gilbarg, D. Some hydrodynamic applications of function theoretic properties of elliptic equations. *Math. Z.* 72 (1959/60), 165-174.

The author establishes some properties of quasi-conformal mapping to prove the following comparison theorem. Let two subsonic flows past the walls  $\gamma$  and  $\bar{\gamma}$  be defined in regions  $D$  and  $\bar{D}$ . Suppose that  $D \subset \bar{D}$  and that  $\gamma$  and  $\bar{\gamma}$  have a regular point  $P$  in common. Then if the respective free stream velocities satisfy the inequality  $q_0 \leq \bar{q}_0$ , the flow speeds at  $P$  are unequal in the same order,  $q(P) \leq \bar{q}(P)$ ; furthermore,  $q(P) = \bar{q}(P)$  if and only if the flows are identical and  $D = \bar{D}$ . It is further shown that all results based on the comparison theorem and on the asymptotic behaviour of the free streamlines, in particular the uniqueness of the infinite cavity, now carry over from incompressible to subsonic flows.

L. M. Milne-Thomson (Madison, Wis.)

6295:

Freeman, N. C. On a singular point in the Newtonian theory of hypersonic flow. *J. Fluid Mech.* 8 (1960), 109-122.

A local examination is made of the neighborhood of  $60^\circ$  from the axis of a sphere, where the Newtonian pressure vanishes and higher approximations are singular. Using magnified variables, a local solution is found that joins the Newtonian solution uniformly with a free layer downstream. Practical application is limited by the unrealistic requirement that  $[(\gamma - 1)/(\gamma + 1)]^{3/11}$  be small.

M. D. Van Dyke (Stanford, Calif.)

6296:

**Maull, D. J.** Hypersonic flow over axially symmetric spiked bodies. *J. Fluid Mech.* 8 (1960), 584-592. (2 plates)

A needle protruding ahead of a blunt body reduces its drag by inducing a conical tip of dead separated air. However, if the flow cannot subsequently follow the body, oscillatory motion occurs. Instantaneous shadowgraph photographs of the cycle are analyzed for a family of bodies at Mach number 6.8.

*M. D. Van Dyke (Stanford, Calif.)*

6297:

**Bloom, Martin H.; Steiger, Martin H.** Inviscid flow with nonequilibrium molecular dissociation for pressure distributions encountered in hypersonic flight. *J. Aerospace Sci.* 27 (1960), 821-835, 840.

Steady one-dimensional inviscid flows of a binary gas mixture involving dissociation and recombination reactions are studied for several assumed pressure distributions. These are generally typical of distributions met with in hypersonic flight. Numerical results are computed for conditions likely to occur in the atmosphere between altitudes of 154,000 ft. and 246,000 ft. and flight velocities between 15,000 and 25,000 ft./sec. The paper thus gives a useful account of conditions likely to be met with in practice.

*J. F. Clarke (Cranfield)*

6298:

**Legendre, Robert.** Analogie hydraulique pour l'étude des écoulements hypersoniques. *C. R. Acad. Sci. Paris* 250 (1960), 3771-3772.

6299:

**Korobeinikov, V. P.; Ryazanov, E. V.** Solutions of singular cases of point explosions in a gas. *Prikl. Mat. Meh.* 23 (1959), 384-387 (Russian); translated as *J. Appl. Math. Mech.* 23, 539-544.

The problem of point explosion in a medium of variable density, solved in the general case by Sedov [*Similarity and dimensional methods in mechanics*, Academic Press, New York, 1959; MR 21 #6840] is considered further. If  $r$  is the radial distance, the density is initially assumed to obey a power law  $r^{-\gamma}$  and the medium to be a perfect gas with constant specific heat ratio  $\gamma$ . The formula for the general solution has singularities for certain values of  $\gamma$  (expressed in terms of  $\omega$ ), and these are investigated separately. It is found that, for fixed  $\omega$ , the solution is continuous in  $\gamma$ , and in particular does not exhibit unusual behavior when  $\gamma \rightarrow 2$ . *M. Holt (Berkeley, Calif.)*

6300:

**Smith, W. R.** Mutual reflection of two shock waves of arbitrary strengths. *Phys. Fluids* 2 (1959), 533-541.

This paper reports on experiments performed in air on the mutual reflection of pairs of unequal and equal plane shock waves. The results for the latter reflections are compared to those obtained in the reflection of plane shocks from rigid walls. Both regular reflection and Mach reflection is observed. The former type seems to terminate at or possibly a little before the extreme angle given by the theory based on the usual Rankine-Hugoniot equations

and the conditions that the pressure behind the shocks is continuous as is the angle of deflection of the flow.

*A. H. Taub (Urbana, Ill.)*

6301:

**Guess, Arnold W.** Density compression ratio across relativistic-strong-shock waves. *Phys. Fluids* 3 (1960), 697-705.

The author assumes that a shock whose velocity compared to the velocity of light is not small is propagated into a medium whose specific enthalpy is small compared to the rest energy. It is also assumed that the radiation energy is negligible compared to the rest energy. The relativistic Rankine-Hugoniot equations are then used to determine the density compression for a material gas, a radiation gas and a material relativistic gas. It is shown that for the material gas the rest density compression ratio increases above its non-relativistic strong shock limit by a term proportional to the square of the ratio of the shock velocity to light velocity.

The velocity of a relativistic sound wave propagating into a mixture of a thermally perfect material gas and a radiation gas is also determined.

*A. H. Taub (Urbana, Ill.)*

6302:

**Rościszewski, Jan.** Calculations of the motion of non-uniform shock waves. *J. Fluid Mech.* 8 (1960), 337-367.

The type of approximate method developed by Chisnell [same *J.* 2 (1957), 286-298; MR 19, 206] and Whitham [ibid. 4 (1958), 337-360; MR 20 #593] for calculating the propagation of shock waves through regions of non-uniform area and flow is extended here to include problems where the shock strength is also modified by interaction with waves overtaking it. In this extension, additional approximations are introduced. The major part of the paper is then devoted to an interesting and varied selection of applications and comparison with results obtained by the numerical method of characteristics. The problems include examples of shocks and detonation waves propagating down non-uniform tubes, interaction with rarefaction waves and contact discontinuities, and analogous problems in steady two-dimensional flow. In all cases the relatively simple approximate calculations agree well with the more accurate solutions.

*G. B. Whitham (Cambridge, Mass.)*

6303:

**Germain, Paul.** L'équation de Burgers et ses applications à la théorie des ondes de choc. *Cahiers de Phys.* 14 (1960), 285-299.

This is an expository article (for the purposes of a conference) on the competition between non-linearity and viscosity in wave motion and shock wave problems, using Burgers's equation  $u_t + uu_x = \nu u_{xx}$  as the basis for the discussion. A full account is available in M. J. Lighthill's contribution in *Surveys in mechanics*, pp. 250-351 (University Press, Cambridge, 1956; MR 17, 1024).

*G. B. Whitham (Cambridge, Mass.)*

6304:

**Sakurai, Akira.** On the problem of a shock wave arriving at the edge of a gas. *Comm. Pure Appl. Math.* 13 (1960), 353-370.

Similarity solutions are found for a plane shock wave

moving through a plane distribution of gas whose undisturbed density is proportional to  $x^\alpha$ . The flow quantities all take the form  $x^\mu f(tx^{-1-\lambda})$ ; the similarity variable  $tx^{-1-\lambda}$  is the same for all quantities and  $\mu$  is a simple expression in terms of  $\alpha$  and  $\lambda$  for each. The shock velocity is proportional to  $x^{-\lambda}$ . The determination of  $\lambda$  is analogous to the determination of the exponent in Guderley's solution for a converging cylindrical shock. There is an unacceptable singularity in the flow (on the limiting characteristic through  $x=0$ ) unless  $\lambda$  takes a certain value; the value depends on  $\alpha$ . For  $\gamma=1.4$ , the values are  $\lambda=0.39334, 0.20214, 0.10352$  for  $\alpha=2, 1, 0.5$ , respectively.

A similarity solution of the same type is used to describe the expansion of the gas after the shock reaches the edge  $x=0$ . Due to the idealization of the continuum theory, the shock imparts an infinite velocity to the edge of the gas and the front then moves to infinity instantaneously. Accordingly there is some difficulty in application of this second stage but overall quantities such as the mass of gas ejected may be useful.

G. B. Whitham (Cambridge, Mass.)

6305:

Cerný, G. G. Application of integral relationships in problems of propagation of strong shock waves. Prikl. Mat. Meh. 24 (1960), 121-125 (Russian); translated as J. Appl. Math. Mech. 24, 159-165.

This is an approximate method for calculating the motion of strong shock waves produced by expanding 'pistons' with plane, cylindrical or spherical symmetry. If the density ratio  $\varepsilon$  from front to back is small, the shock remains close to the piston, the fluid velocity is approximately constant and equal to its value at the shock, and the pressure difference between the shock and the piston is the mass of fluid between the two multiplied by the acceleration of the fluid. All these quantities can be easily calculated in terms of the motion of the shock. Up to this point there is only a crude approximation available for the shock motion: it is exactly the same as the given piston motion. In principle this is improved by going to higher terms in  $\varepsilon$ . However, a better overall approximation is obtained by equating the energy in the fluid to the work done by the piston, and using this relation to determine the shock motion in terms of the piston motion. The approximate results are checked by comparison with known similarity solutions such as the point blast explosion and the solution for a uniformly expanding sphere. The comparison is satisfactory for  $\varepsilon$  up to about 0.2 or 0.3 corresponding to pressure ratios of 5 to 7 for air (assumed polytropic).

G. B. Whitham (Cambridge, Mass.)

6306:

Iyahov, G. M.; Polyakova, N. I. Approximate method of calculation of shock waves and their interaction. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. 1959, no. 2, 13-18. (Russian)

Consider the Lagrangian form of the equations of non-steady one-dimensional flow. Neglect the variation of entropy, and approximate pressure by a polygonal function of specific volume, with  $p = -A_j^2 V + B_j$  on a typical segment. Then the pressure and velocity satisfy one-dimensional wave equations with solutions  $p = F_1(h - A_j t) + F_2(h + A_j t)$  and  $A_j u = F_1 + F_2$ . The author determines the approximate motion of a shock advancing into

stagnant gas as the result of applying at location  $h=0$  a pressure that is a polygonal function of time. By imposing at the shock only the conservation of mass and momentum, and by exploiting the empirical result that  $F_2$  is approximately constant there, he is able to determine  $F_1(h - A_j t)$  and also the motion of the shock  $h = h_0(t)$ . The complete solution involves alternately patching together regions of variable and constant states. Shock reflection at a rigid wall is also considered.

J. H. Giese (Aberdeen, Md.)

6307:

Slėzkin, N. A. Theory of gas flow in the layer between a shock wave surface and a blunt body of revolution. Izv. Akad. Nauk. SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. 1959, no. 2, 3-12. (Russian)

In the narrow layer between a blunt body and a detached shock wave introduce curvilinear coordinates, with  $x$  along the body, and  $y$  orthogonal thereto. Let  $y=0$  on the body,  $y=h(x)$  on the shock, and let  $\varepsilon$  be the mean value of  $h/l$  on an initial portion of the body of length  $l$ . For any section  $x=x_0$  choose as reference values  $U_B, p_B, \rho_B, T_B$  at the point  $y_B$  where  $u(x_0, y)$  assumes its maximum  $U_B$ . With these reference values write in dimensionless form the equations of steady compressible viscous flow with heat conduction, and stretch  $y$  and  $v$  by a factor  $1/\varepsilon$ . At great enough distances downstream, where Reynolds, Mach, and Prandtl numbers  $R_B \sim 1/\varepsilon^2$ ,  $M_B \sim 1$ , and  $P_B \sim 1$ , retention of the lowest order terms in  $\varepsilon$  yields Prandtl boundary layer equations. Near the tip of the body where  $R_B \sim 1/\varepsilon$ ,  $M_B \sim \varepsilon^2$ , and  $P_B \sim 1$  the lowest order terms yield instead Reynolds lubrication equations. The author linearizes the latter system by expanding flow functions about their values at  $B$  as power series in  $\varepsilon$ , substituting, and retaining only first order terms. The linearized system is supplemented by boundary conditions on body and shock, with shock conditions that take account of viscosity and heat conduction. The linearized Reynolds equations can be integrated explicitly, to the extent that the shock boundary conditions and the definition of  $y_B$  yield a system of eight finite and ordinary differential equations for  $h(x)$ ,  $U_B$ ,  $T_B$ , and the values of the five flow functions immediately behind the shock. Limiting values at the intersection of shock wave and axis of symmetry are also considered.

J. H. Giese (Aberdeen, Md.)

6308:

Merk, H. J. Analysis of heat-driven oscillations of gas flows. I. General considerations. Appl. Sci. Res. A 6 (1957), 317-336.

Author's summary: "It has been known for a long time that gas flows through systems containing heat sources can produce acoustic oscillations which are coupled with a periodic heat release of the heat source (heat-driven oscillations). This phenomenon may at times occur in industrial combustion systems and in jet engines, and can under certain circumstances be very awkward owing to the intensity and shrillness of the noise, sometimes also owing to the actual damage caused by the pressure fluctuations, and to the reduced performance of the combustor or jet engine. In this paper an effort is made to create a basis for the theoretical treatment of heat-driven oscillations. The fundamental ideas underlying the theory are discussed with reference to a simple model consisting

of a tube containing a disk-shaped heat source perpendicular to the axis of the tube. Upstream of the heat source the oscillations are mainly acoustic, but downstream of the heat source there is not only an acoustic mode but also an entropy mode of the oscillations. It is proved that the existence of the entropy mode can be neglected if the effects of the viscosity and Mach number on the oscillations are slight, so that quadratic and higher terms in the viscosity and Mach number can be neglected. Under these circumstances the characteristic equation governing the occurrence of heat-driven oscillations has to be derived from the laws of conservation of momentum and energy applied across the heat source, since it appears that these laws contain only acoustic quantities. In order to obtain the characteristic equation in a general form, transfer functions of the heat source and acoustic admittances and impedances are introduced. The transfer functions of the heat source describe the response of the heat release to fluctuations in the velocity and thermodynamic variables of the gas flow, and in some forthcoming papers will be considered in more detail for hot wire gauzes and flames of premixed gases."

6309:

Merk, H. J. Analysis of heat-driven oscillations of gas flows. II. On the mechanism of the Rijke-tube phenomenon. *Appl. Sci. Res. A* 6 (1957), 402-420.

Author's summary: "The case treated is that of a straight tube open at both ends and provided with a hot wire gauze composed of a thin spirally wound metallic wire. The general form of the characteristic equation governing the excitation of sound when air is forced to flow through the tube was derived in a previous paper. A discussion now follows of the transfer functions of the hot wire gauze and the dissipation of acoustic energy in the tube; it is shown that a first-order solution of the characteristic equation can be obtained which accounts for the main features of the Rijke-tube phenomenon, viz. that excitation of the fundamental frequency of the tube occurs only when the hot wire gauze is in the entrance half of the tube, and that maximum excitation occurs when it is located at about one fifth of the tube length from the entrance. Neutral curves for the fundamental frequency are calculated which show that for excitation to occur the Strouhal number of the hot wire should have a value within a critical range. This means that to any given dimensions of the tube and the wire gauze there corresponds a critical range of values of the mean velocity of the gas flow outside which there should be no sound generation. Finally neutral curves for the first overtone are presented which show that there are two positions of the wire gauze that give rise to excitation of this harmonic."

6310:

Westervelt, Peter J. Effect of sound waves on heat transfer. *J. Acoust. Soc. Amer.* 32 (1960), 337-338.

From the author's summary: "The suggestion is put forward that the effect of sound waves on heat transfer is dominantly a result of the modification of the inner streaming boundary layer which is known to occur when the sound particle displacement amplitude  $s$  exceeds in magnitude the acoustic boundary layer thickness  $\delta_{ac}$ ."

J. W. Miles (Los Angeles, Calif.)

6311:

McLachlan, N. W. ★Loud speakers: Theory, performance, testing and design. Corrected ed. Dover Publications, Inc., New York, 1960. xii + 399 pp. \$2.25.

This is an unabridged and corrected version of the first edition published in 1934 [Clarendon Press, Oxford].

6312:

Brigham, G. A.; Borg, M. F. An approximate solution to the acoustic radiation of a finite cylinder. *J. Acoust. Soc. Amer.* 32 (1960), 971-981.

The radiation field of a finite cylinder whose lateral surface is vibrating in a normal mode is approximated by calculating the field of a corresponding surface distribution of point sources in a homogeneous medium. A second approximation, taking some account of the presence of the cylinder, is obtained by including the contribution of only that half of the surface which faces in a given direction. A third calculation utilizes the solution for a strip source on an infinite cylinder. Many numerical calculations of the radiation patterns are given for various modes of vibration, ratio of dimensions to wavelength ( $0.3 < ka < 10$ ), and presence or absence of plane reflecting boundaries.

The results of the first two methods are very similar; the third shows considerable discrepancy, although all three not surprisingly agree at higher frequency. Comparison is made with some experimental results, and the lack of agreement is ascribed to non-ideal experimental conditions.

{The conclusion that the agreement between the first two types of approximation for a cylinder justifies their use in problems involving finite sources of other geometries is rather dubious; indeed it may be that they are both in error even for the cylinder.}

E. T. Kornhauser (Providence, R.I.)

6313:

Heaps, H. S. General theory for the synthesis of hydrophone arrays. *J. Acoust. Soc. Amer.* 32 (1960), 356-363.

A mathematical approach is given for optimizing both the volume distribution of a continuous array of hydrophone elements and the linear combination of their outputs in order to produce a single instantaneous signal peak exceeding as much as possible the root-mean-square background noise. The (non-harmonic) signal and the mean background noise power are expressed via the convolution theorem as functions of a set of independent complex parameters involving position and frequency. Differentiation of the signal-to-noise ratio with respect to these parameters leads to optimum design values for the distribution and frequency responses of the hydrophone elements in a given volume.

W. W. Soroka (Berkeley, Calif.)

6314:

Fixman, Marshall. Ultrasonic attenuation in the critical region. *J. Chem. Phys.* 33 (1960), 1363-1370.

Author's summary: "The interaction of forced sound waves with critical density fluctuations is discussed quantitatively. The nonlinear equations of motion are analyzed to distinguish three contributions to the instantaneous density and temperature variation: (1) the

spontaneous fluctuations that are defined as the solution to an initial value problem, (2) the macroscopically observed sound wave which oscillates with harmonic time dependence, and (3) the result of an interaction between (1) and (2). The three contributions satisfy three equations, which are discussed separately. A completely general combination of these equations to compute ultrasonic absorption and attenuation is avoided, as the requisite thermodynamic and transport coefficients are inadequately known. Because, however, a thermal relaxation mechanism has previously been treated on the basis of an oversimplified multiphase picture of spontaneous fluctuations, the present calculations are carried to completion on the assumption that only the local heat capacity is affected by the density fluctuations. The sound wave together with the fluctuating heat capacity produce temperature fluctuations, whose relaxation is the source of the absorption."

6315:

Fritsche, L. Theorie des akustischen Zylinderresonators unter Berücksichtigung der Schallanregung. II. *Acustica* 10 (1960), 199-207. (English and French summaries) [Part one (experimental) is in *Acustica* 10 (1960), 189-198.]

Author's summary: "From the laws of conservation of fluid mechanics and the equation of state of ideal gases, an approximate solution of the boundary value problem for a closed, gas-filled cylindrical resonator is deduced considering the excitation of the sound. The quality of this approximation agrees with the requirements generally given by the order of magnitude of experimental error ( $> 1\%$ ). Infinite thermal inertia and rigid walls as well as infinitely small sound amplitudes are prerequisites of the theory. Of the transportation phenomena only heat conduction and friction are taken into account. Relaxation processes in gases are not taken into consideration. The linear extension of the theory to such processes is possible without difficulties by the introduction of a time dependent dynamic equation of state.

"The theory is in very good agreement with the experimental results reported in a preceding work."

6316:

Kawasima, Y. Sound propagation in a fibre block as a composite medium. *Acustica* 10 (1960), 208-217. (French and German summaries)

Author's summary: "The sound absorbing properties of a fibre block are treated with a composite medium model, in which the fibres are suspended in the air under certain binding forces. The sound absorption is considered to be caused by the frictional losses arising from the relative velocities and the heat conduction between the air and fibres. In usual cases, the latter effect is found to be small compared with the former ones. The dependence, in a microscopic sense, of the acoustic characteristics on the geometrical structures and the physical characteristics of the fibre block are derived theoretically."

6317:

Nomura, Yûkichi; Yamamura, Ichirô; Inawashiro, Sakari. On the acoustic radiation from a flanged circular pipe. *J. Phys. Soc. Japan* 15 (1960), 510-517.

The problem of pure tone radiation from an infinitely long pipe into the half-space formed by an infinite flange at the pipe's open end is solved using Weber-Shafheitlin type integrals and Jacobi's polynomials. Scalar wave functions are set up in cylindrical coordinates inside the pipe and in the half-space. Continuity of the wave function and of its axial derivative across the pipe opening yields two equations for evaluating the unknown coefficients by expansion formulas. Numerical results are presented as functions of pipe radius to acoustic wavelength up to a ratio of about 0.6. These include plots of radiation resistance and reactance, directivity, transmission and reflection coefficients and end correction. Comparisons are made with Levine-Schwinger results for an unflanged pipe.

W. W. Soroka (Berkeley, Calif.)

6318:

Butrov, M. V. Diffraction of a scalar wave by a slit and by a circular aperture in a screen of arbitrary thickness. *Akust. Zh.* 6 (1960), 16-22 (Russian); translated as *Soviet Physics. Acoust.* 6, 13-19.

The author considers the problem of determining the transmission coefficient of a plane, monochromatic, scalar (acoustic) wave normally incident on an endless slit or a circular hole in a perfectly soft screen of arbitrary thickness. Expanding in waveguide modes the potential function in the opening and expressing the potential functions to the left and right of the screen in terms of the aperture potentials  $\psi^{(-)}$  and  $\psi^{(+)}$  at the two end surfaces of the opening, he casts the problem in the form of a dual set of integral equations involving  $\psi^{(-)}$  and  $\psi^{(+)}$ . These equations he reduces to a pair of mutually independent integral equations, one involving only the difference  $\psi^{(-)} - \psi^{(+)} = \psi_1$  and the other only the sum  $\psi^{(-)} + \psi^{(+)} = \psi_0$ . From the integral equations for  $\psi_0$  and  $\psi_1$  he deduces a variational principle for the transmission coefficient. He carries out numerical computations for the case of the circular hole.

C. H. Papas (Pasadena, Calif.)

6319:

Rathna, (Miss) S. L. Superposability of steady axisymmetrical flows in a non-Newtonian fluid. *Proc. Indian Acad. Sci. Sect. A* 51 (1960), 155-163.

Using a special case of the Reiner-Rivlin theory of fluids the author finds conditions under which two axially symmetric velocity fields are superposable. By definition, two velocity fields are superposable if each and their sum is consistent with the equations of motion.

J. L. Ericksen (Baltimore, Md.)

6320:

Hasimoto, Hidenori. Steady longitudinal motion of a cylinder in a conducting fluid. *J. Fluid Mech.* 8 (1960), 61-81.

This paper concerns steady motion of an infinitely long, solid cylinder parallel to its axis (the  $z$  axis) in an unlimited body of electrically conducting, incompressible, viscous fluid. There is a magnetic field that is uniform and parallel to the  $y$  axis at large distances from the cylinder. The problem is formulated for arbitrary values of the electric conductivity of the cylinder and then specialized for the special cases of perfectly conducting and perfectly insulating cylinders. It is shown that the disturbance fields are made up of solutions of the Oseen equation:

thus these are parabolic "wakes" extending along the  $y$  direction at large values of  $|y|$ . There is a simple relation between the drag per unit length of the cylinder and the difference of electric potential at  $x = +\infty$  and  $x = -\infty$  for fixed  $y$ . These quantities are also related to the strength of the wakes at  $|y| = \infty$ . The problem is simplified for an insulating cylinder, since it reduces to a more familiar boundary-value problem.

As examples, the cases of insulating and perfectly conducting flat plates are treated. Drag is calculated in detail. If the plate is semi-infinite in width there are simple solutions which are approximations for finite-width plates at large Hartmann number. Near an insulating plate there is a boundary layer; a perfectly conducting plate, on the other hand, carries with it most of the fluid above and below it. From these results, a description of flows about other cylinders at large Hartmann number is deduced. Finally, there is some discussion about the flow about cylinders of finite conductivity.

W. R. Sears (Ithaca, N.Y.)

6321:

Kogan, M. N. Plane flow of a perfectly conducting fluid with the magnetic field nearly parallel to the velocity. Dokl. Akad. Nauk SSSR 130 (1960), 284-286 (Russian); translated as Soviet Physics. Dokl. 5, 40-42.

The author considers the two-dimensional flow of a perfectly conducting fluid in a magnetic field past a cylindrical dielectric obstacle, with special reference to the boundary layer. The pressure is assumed constant.

A. Herzenberg (Manchester)

6322:

Skuridin, G. A.; Stanyukovich, K. P. The motion of a conducting plasma under the action of a piston. Dokl. Akad. Nauk SSSR 131 (1960), 72-74 (Russian); translated as Soviet Physics. Dokl. 5, 283-286.

In an earlier paper in same Dokl. 130 (1960), 1248-1251 [MR 22 #4349], the authors gave a formal asymptotic solution of the non-linear system of partial differential equations which describe the one-dimensional motion of a non-viscous, finitely conducting plasma. The solution given in this paper contains arbitrary functions arising from integrations. The authors asserted that the arbitrary functions could be determined in a specific problem when the boundary conditions are given. In the present paper the authors treat the motion of the plasma under the action of a piston and show how the arbitrary functions can be determined for this situation. The important quantities about the motion of the shock wave are obtained.

M. Kline (New York)

6323:

Carini, Giovanni. Considerazioni energetiche in magnetoidrodinamica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 27 (1959), 48-53.

6324:

Karplus, Robert. Radiation of hydromagnetic waves. Phys. Fluids 3 (1960), 800-805.

The author studies the radiation of hydromagnetic waves in a uniform, fully ionized, perfectly conducting, pressureless fluid. These waves are supposed to be excited by a

varying current injected into the fluid. The phenomena in general has been previously studied but in this article the author actually constructs the Green's function for a special, simple case explicitly and applies it to simple models of the current distributions, such as magnetic dipole radiation, etc. As usual in these problems, the method of linear approximations is used. This is permissible if the field and density fluctuations are very small. The author finds that if the source contains a current along the constant static magnetic field, then the radiation along the field lines through the source is very intense. This is also true in the case of a non-uniform static magnetic field of an infinitely long straight wire.

R. S. B. Ong (Leiden)

6325:

Braginskii, S. I. Magnetohydrodynamics of weakly conducting liquids. Zh. Eksper. Teoret. Fiz. 37 (1959), 1417-1430 (Russian. English summary); translated as Soviet Physics. JETP 10 (1960), 1005-1014.

The magnetohydrodynamic equations, simplified by treating the magnetic Reynolds number as very small, are applied to a variety of soluble special cases of flow in an external magnetic field. These cases include: (1) plane waves and surface waves; (2) flow at large Hartmann numbers along tubes of arbitrary cross-section, the walls being conductors, insulators, or a combination of the two with the separate conducting parts connected by external resistors; also one wall may be replaced by a free surface; (3) rotation caused either by the motion of an insulating or conducting bottom of the container or by a radial current.

O. Penrose (London)

6326:

Carstoiu, John. Hydromagnetic waves in a compressible fluid conductor. Proc. Nat. Acad. Sci. U.S.A. 46 (1960), 131-136.

The author considers the propagation of small disturbances in an infinite continuum of an infinitely conducting fluid at rest in the presence of an initial uniform magnetic field  $H_0$  concentrating on the propagation of vorticity field  $\omega$  and current density field  $j$  rather than on the velocity field and the magnetic field. In the case of a compressible fluid, only the components of vorticity and current density in the direction of  $H_0$  are propagated with Alfvén velocity  $a$  and without damping. In the special case when  $\omega$  is parallel to  $H_0$  taken along the  $z$ -axis, the author discusses a mechanical model of the propagation of vorticity in the direction of  $H_0$  in a system of  $n$  charged particles placed on the  $z$ -axis at equal intervals of distance  $d$ . In the opinion of the reviewer the conclusion that the wave is propagated only if the initial frequency of disturbance is less than twice the reciprocal of the time taken by the disturbance to travel from one particle to the next is based on the discrete particle model of plasma for which the continuum mechanics is not valid. This view is supported by the following considerations: (1) the equation governing the propagation of vorticity is the undamped wave equation so that given sufficient time, the disturbance will reach any point in the system; (2) in the case of continuum  $d \rightarrow 0$  and consequently even under the above condition, the propagation will be possible for all frequencies.

P. L. Bhatnagar (Bangalore)

6327:

Tarasov, Yu. A. On the stability of plane Poiseuille flow of a plasma of finite conductivity in a magnetic field. *Ž. Èksper. Teoret. Fiz.* **37** (1959), 1708-1713 (Russian. English summary); translated as Soviet Physics. JETP **10** (1960), 1209-1212.

This article is intended to fill the gap between the problem of the stability of a fluid with infinite conductivity treated by Velikhov [same *Ž.* **36** (1959), 1192-1202; MR **21** #7679] and the stability of a fluid with poor conductivity discussed by Stuart [Proc. Roy. Soc. London. Ser. A **221** (1954), 189-206; MR **15**, 907]. In other words, the author treats the case where  $R_m \leq 1$ , in which  $R_m$  is the magnetic Reynolds number. The analysis used is similar to that employed in the ordinary hydrodynamic case and it provides a series of neutral curves for various values of  $R_m$ . For  $R_m = 1$  the results indicate that the square of the critical Alfvén number ( $A_{crit}^2 = B_0^2/u^2 \cdot 4\pi\rho$ ) lies between 0.12 and 0.13, which means that this is about 3.5 times greater than the  $A_{crit}$  for a plasma of infinite conductivity. It is also shown that when  $R_m$  is increased from  $R_m \sim 3$  or 4 to  $R_m = \infty$  the magnitude of  $A_{crit}$  varies only slightly. A curve is given showing the dependence of  $A_{crit}$  on  $R_m$ . R. S. B. Ong (Leiden)

6328:

Čekmarev, I. B. Nonstationary flow of a conducting fluid in a flat tube in the presence of a transverse magnetic field. *Ž. tehn. Fiz.* **30** (1960), 338-344 (Russian); translated as Soviet Physics. Tech. Phys. **5**, 313-319.

The author obtains an exact solution for the particular case of non-stationary flow of a conducting, incompressible, viscous fluid between parallel conducting walls of infinite length in the presence of a transverse magnetic field. The method of Laplace transform is employed in the analysis to arrive at expressions for the fluid velocity and the intensities of electric and magnetic fields as a function of the conductivities of the fluid and the walls of the tube. A complete solution to the problem is obtained in the particular case when the walls are ideally conducting. R. S. B. Ong (Leiden)

6329:

Kendall, P. C. Hydromagnetic oscillations of a rotating liquid sphere. *Quart. J. Mech. Appl. Math.* **13** (1960), 285-299.

The paper deals with the problem of the oscillations of a rotating liquid sphere, which is infinitely conducting, in a uniform magnetic field parallel to the axis of rotation. The angular velocity is assumed to be so small that any deviation of the surface from the spherical form could be neglected. The equation governing the hydromagnetic oscillations is solved in cylindrical coordinates. The boundary conditions are such that the excess pressure vanishes at the free surface of the liquid, and the magnetic field is taken to be continuous at the boundary. This leads to three infinite sets of equations for three infinite sets of unknowns. The period equation is obtained in the form of an infinite determinant. Numerical approximations for the period, in the case when the equatorial velocity is much larger than the Alfvén velocity, for the even and odd fundamental modes have been obtained up to the sixth order. The convergence of the series obtained is not

rapid. However the values obtained for the period confirm an earlier order of magnitude of the period as obtained by Cowling [Proc. Roy. Soc. London. Ser. A **233** (1955), 319-322; MR **17**, 904]. F. C. Auluck (Delhi)

6330:

Davies, T. V. On steady axially symmetric solutions of the idealized hydromagnetic equation for a compressible gas in which there is no diffusion of vorticity, heat, or current. *Quart. J. Mech. Appl. Math.* **13** (1960), 168-183.

The author discusses the steady axially symmetric flow of a compressible electrically conducting gas. In his discussion he assumes that the dissipative mechanism, such as viscosity, thermal conductivity and electrical resistance, are absent. It turns out that the fields of velocity, magnetic induction and electric current are expressible in terms of two functions of position. These functions are further related by two differential equations. In the course of his investigation of these equations the author discovers certain singular surfaces. The property of these surfaces is that in their neighborhood, the streamlines and the lines of induction tend to become parallel.

The paper ends with the solutions of some special flow configurations. Ram P. Kanwal (University Park, Pa.)

6331:

Kendall, P. C. The variational formulation of the magneto-hydrostatic equations. *Astrophys. J.* **131** (1960), 681-683.

The author obtains all the solutions of the equations of magnetohydrostatic equilibrium of an infinitely conducting gaseous system with the help of the variational principle by considering the stationary values of the potential energy

$$V = \int \left[ \frac{H_i H_i}{8\pi} + \frac{p}{\gamma - 1} \right] d\tau,$$

where  $H_i$  are the components of magnetic field,  $p$  the pressure in the gas,  $\gamma$  the ratio of specific heats, and the integration extends over the entire volume of the configuration. In the proof the author makes use of the integral of the induction equation  $H_i = a_{ij} H_j^{(0)} / \Delta$ , where the superscript (0) denotes the initial state of the system,  $a_{ij} = \partial x_i / \partial x_j^{(0)}$  are the elements of the matrix of transformation which brings the fluid element initially at  $x_j^{(0)}$  to the place  $x_i$ , and  $\Delta = |a_{ij}|$ . In the present problem, it is not necessary to assume some additional conservation laws like the conservation of the total angular momentum of the system as was necessary in the work of Woltjer [Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 489-491, 833-841; **45** (1959), 769-771; *Astrophys. J.* **130** (1959), 400-404, 404-413; MR **20** #3025, 2968; **22** #1291, 1292, 1293] and Chandrasekhar [Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 842-847; MR **20** #3026].

P. L. Bhatnagar (Bangalore)

6332:

Regier, S. A. Concerning the thermal effect in the flow of an electrically conducting fluid between parallel walls. *Prikl. Mat. Meh.* **23** (1959), 948-950 (Russian); translated as J. Appl. Math. Mech. **23**, 1346-1350.

Steady parallel flow of an incompressible viscous fluid (as in Hartmann's problem) is considered. Only the case

of zero total current flow is treated. The generalization introduced here is that the coefficient of viscosity may be a function of temperature; however, the heat conductivity and electrical conductivity are assumed constant. Temperature profiles are calculated for Hartmann flow (constant viscosity) with consideration of Joule and frictional heating, separately. With variable viscosity, series solutions must be resorted to; here a certain temperature dependence is chosen and the solution in series is discussed, but no results are given.

W. R. Sears (Ithaca, N.Y.)

6333:

Kulikowski, A. G.; Lyubimov, G. A. Gas-ionizing magnetohydrodynamic shock waves. Dokl. Akad. Nauk SSSR 129 (1959), 52-55 (Russian); translated as Soviet Physics. Dokl. 4 (1960), 1185-1188.

The authors treat the problem of determining the intensity of plane magnetohydrodynamic shock waves and their generated electromagnetic waves in a gas unionized ahead of the shock front. They restrict attention to two cases in which only two entropy producing processes are important: (1) magnetic and molecular viscosity or (2) magnetic viscosity and heat conductivity. In each of these cases they also assume that the electric and magnetic fields are perpendicular to each other and to the direction of propagation.

They begin the analysis by writing the system of first order nonlinear ordinary differential equations appropriate to the two cases for an ideal gas. Then qualitative aspects of the integral and boundary condition curves in the  $v$ - $H$  (phase) plane are considered for the regions where the conductivity is greater than zero and then where it is zero. The properties of the solutions are studied relative to the nature of the singular points and a variety of additional assumptions on the transition zone. In this discussion,  $v$  has been replaced in the translation by  $\nu$ , on p. 1186, lines 11, 18, 22, 24 and 25. (Another error in translation: the paper should end with the words "are known".) The authors are able to derive the intensity of the coherent electromagnetic wave in terms of the velocity of the associated shock wave, and then outline the solution method for obtaining the shock velocity in terms of the piston velocity.

J. E. Drummond (Seattle, Wash.)

6334:

Karol', I. L. The influence of turbulent diffusion in the direction of the wind on the distribution of concentration of a substance diffusing in the atmosphere. Dokl. Akad. Nauk SSSR 131 (1960), 1283-1286 (Russian); translated as Soviet Physics. Dokl. 5, 264-268.

The author considers diffusion from an elevated point source. In the case of constant coefficients for the turbulent diffusion it is shown that the effects due to the coefficient for horizontal diffusion in the wind direction can not be neglected. In contrast, when the coefficient for vertical diffusion is proportional to height, it is shown that the down-wind coefficient can have negligible effect on the ultimate concentration distribution.

W. V. R. Malkus (Woods Hole, Mass.)

6335:

Aidi, Osmane. ★Contribution à l'étude des diffuseurs courts de révolution. Publ. Sci. Tech. Ministère de l'Air, No. 362, Paris, 1960. x+106 pp. 21.10 NF.

6336:

Horvay, G. Flow of liquid metal past a porous flat plate. J. Appl. Mech. 27 (1960), 749-750.

# OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 6112, 6324, 6577, 6578, 6586.

6337:

Buchdahl, H. A. Optical aberration coefficients. III. The computation of the tertiary coefficients. J. Opt. Soc. Amer. 48 (1958), 747-756.

[For part II, see same J. 48 (1958), 563-567; MR 20 #2972.]

A computing scheme is developed for the full set of primary, second, and tertiary (orders 3, 5, and 7, respectively) monochromatic aberration coefficients of symmetrical optical systems composed of spherical surfaces. The notation and concepts are those of the author's monograph, *Optical aberration coefficients* [Oxford Univ. Press, London, 1954; MR 19, 354]. The computing scheme is intended for an ordinary desk calculating machine, and a somewhat different scheme would be appropriate if an electronic computer is to be used.

G. L. Walker (Providence, R.I.)

6338:

Buchdahl, H. A. Optical aberration coefficients. IV. The coefficient of quaternary spherical aberration. J. Opt. Soc. Amer. 48 (1958), 757-759.

As a sequel to the first paper of the present series, in same J. 46 (1956), 941-943 [MR 19, 355], devoted to the development of a computing scheme for the coefficient of tertiary (order 7) spherical aberration, a computing scheme is now developed for the coefficient of quaternary (order 9) spherical aberration.

G. L. Walker (Providence, R.I.)

6339:

Rosendahl, Gottfried R. Contributions to the optics of mirror systems and gratings with oblique incidence. I. Ray tracing formulas for the meridional plane. J. Opt. Soc. Amer. 51 (1961), 1-3.

Ray tracing formulas are developed for decentered optical systems involving mirrors and gratings.

E. W. Marchand (Rochester, N.Y.)

6340:

Barikowski, Z. Aplanatic lens and two-mirror aplanat. Bull. Soc. Amis Sci. Lettres Poznań. Sér. B 15 (1958/59), 161-169 (1960).

6341:

Stavroudis, Orestes N. Lens design: a new approach. J. Res. Nat. Bur. Standards Sect. B 63B (1959), 31-42.

This work deals with the Jacobian of the transformation from the 4-space of object rays to the 4-space of image rays of an optical system. Formulas are developed for the computation of the Jacobian, in a special coordinate system for a rotationally symmetric optical system, as a product of Jacobians associated with the component

refracting surfaces of the system. This leads to the differential equations of perfect rotationally symmetric optical systems with finite object and image planes.

The author's abstract claims: "This paper describes a new method of defining the total aberrations of an optical system and its application to lens design", but the paper contains only two statements on this subject. Referring to the Jacobian: "Its elements consist of the partial derivatives of image space variables with respect to object space variables and therefore can be said to represent the geometrical aberrations of an optical system;" and "a measure of the failure of the equations (referring to the differential equations of a perfect system) to be satisfied can be taken as a measure of the aberrations of the optical system." Perhaps these suggestions are to be developed elsewhere.

G. L. Walker (Providence, R.I.)

6342:

Miyamoto, Kenro. On the design of optical systems with an aspheric surface. *J. Opt. Soc. Amer.* 51 (1961), 21-22.

Author's summary: "This short article is concerned with the problem of designing a correcting aspherical surface which is situated in an arbitrary position within an optical system. The method requires the solution of two first-order differential equations and seems more suitable for the use with an electronic computer than methods previously proposed by other authors. It can be considered as a differential form of a method previously proposed by E. Wolf."

6343:

Schlomka, Teodor. Zur Lichtreflexion am bewegten Spiegel. *Wiss. Z. Hochsch. Architekt. Bauwesen Weimar* 7 (1959/60), 151-154. (Russian, English and French summaries)

Formulas are derived for the change in frequency and direction of rays in connection with reflection at a mirror having an arbitrary direction of movement. It is found that only the velocity component in the direction of the normal need be considered.

E. W. Marchand (Rochester, N.Y.)

6344:

Biot, A. *Optique géométrique et optique ondulatoire. Un point de méthode.* *Ann. Soc. Sci. Bruxelles. Sér. I* 74 (1960), 31-34.

In this note the author discusses briefly the proper domain of application of geometrical optics in the description of light phenomena. He points out that it is only a partial theory, since it describes only the propagation of light without considering the nature of light (frequency, energy), i.e., a theory of propagation of light in terms of rays, based on Fermat's principle. When energy consideration and the structure of images are concerned, one must go beyond the geometrical optics description of light. The author objects to statements which are often made (literally or figuratively), that radiation energy is propagated along rays, since the rays are simply geometric curves and thus must carry infinite energy. Similar objections are raised about point sources, etc.

In this connection the reviewer wishes to remark that

the objections raised by the author are well known and are only employed in a figurative sense. The above objections as well as the limitation of geometrical optics in describing optical phenomena are carefully stated and discussed in modern books in this field [see for instance, M. Born and E. Wolf, *Principles of optics*, Pergamon, New York, 1959; MR 21 #6918; R. K. Luneberg, *Mathematical theory of optics*, Brown Univ., Providence, R.I., 1944; MR 6, 107]. In Luneberg's book one finds a proof that the field vectors  $E$  and  $H$  are propagated along light rays and are orthogonal to the direction of propagation. The discontinuities of  $E$  and  $H$  on a wave front also remain parallel along any light ray in the sense of Levi-Civita. Furthermore, one will also find a clear and detailed presentation of the relationship between geometrical optics based on the Hamiltonian theory and the Maxwellian theory of optics.

N. Chako (New York)

6345:

Kadomcev, B. B. Principle of invariance for a homogeneous medium of arbitrary geometrical form. *Dokl. Akad. Nauk SSSR* 112 (1957), 831-834. (Russian)

A principle of invariance for the scattering of light by a homogeneous medium of convex boundary is obtained. It is assumed that the scattering does not affect the frequency. A transport equation is derived which does not contain an explicit dependence on the radius vector, and is therefore invariant with regard to infinitesimal spatial displacements.

E. Gora (Providence, R.I.)

6346:

Ramachandran, G. N. A problem in probability related to the passage of light through a cloud of particles. *Proc. Indian Acad. Sci. Sect. A* 52 (1960), 87-99.

The problem of finding the energy contrast of a light beam after it has encountered a number of particles of given cross-sectional areas is equivalent to that of finding the probable area covered by a number of discs thrown at random over a given area. Cases are considered in which the sizes, shapes and orientation of the discs are varied.

E. W. Marchand (Rochester, N.Y.)

6347:

Evett, A. A.; Fried, D. C. Speed of light in flowing dispersive liquids. *Amer. J. Phys.* 28 (1960), 733-735.

Authors' summary: "A general, rigorous method is presented for obtaining the equation for the speed of light in a flowing liquid. The derivation leads to an expression correct to all orders in the ratio of the speed of the liquid to the speed of light in a vacuum. The procedure used is based on establishing a differential equation which relates the light frequency to the speed of the liquid. Comparison is made between this general approach and the more usual approach. Limitations of the usual approach are discussed."

6348:

Namioka, Takeshi. Theory of the ellipsoidal concave grating. I. *J. Opt. Soc. Amer.* 51 (1961), 4-12.

Ellipsoidal concave gratings are shown to have theoretical properties superior to concave spherical gratings,

especially with regard to astigmatism and spherical aberration. However, technical problems make the manufacture of ellipsoidal gratings very difficult.

*E. W. Marchand (Rochester, N.Y.)*

6349:

Namioka, Takeshi. Theory of the ellipsoidal concave grating. II. Application of the theory to the specific grating mountings. *J. Opt. Soc. Amer.* 51 (1961), 13-16.

The previously developed theory of ellipsoidal concave gratings [see preceding review] is applied to the Seya-Namioka monochromatic grating and specifications for mounting such a grating in this instrument are described.

*E. W. Marchand (Rochester, N.Y.)*

6350:

Kuščer, I.; Ribarič, M. Matrix formalism in the theory of diffusion of light. *Opt. Acta* 6 (1959), 42-51.

In this article the authors discuss the problem of diffusion of a polarized incident light wave in a slightly absorbing medium. Instead of representing the field vector  $E$  in terms of two linearly polarized states (commonly employed when polarization effects are taken into account), the authors have resolved  $E$  in terms of two circular and oppositely polarized waves. The advantage of this representation is that each component of the vector field  $E$  and of the density matrix  $I$  is transformed independently of the other, when the plane of reference, properly chosen (say the meridian plane), is rotated about the direction of propagation. For an isotropic medium devoid of rotary power, the components of the scattering matrix transform likewise independently by such a rotation, if the scattering plane is taken for the reference plane. For this case the components of the scattering matrix depend only on the angle  $\theta$  between the incident and the scattered directions, with the rotations entering as exponential factors. This makes it possible to use the addition theorem of Legendre polynomials and to expand each component of the scattering matrix in a series of products of Legendre polynomials, each depending only on the respective direction of the incident and scattering beams.

The equation of transfer of energy reduces to a system of four integro-differential equations in the four components of the density matrix, where the scattering matrix (components) is the kernel of the equation. This procedure is applied for the case of a uniform medium of infinite extent with sources located at infinite distances. If the medium is slightly absorbing, their results show that the diffusion length is very large and the angular distribution becomes almost isotropic. The polarization effect disappears altogether in the first two orders of their series solution. For a non-absorbing medium, the diffusion length becomes infinite and, furthermore the solution reduces to that of isotropic distribution with no polarization. For this case one can obtain a more general solution of the transport equation, showing anisotropic angular distribution, but devoid of polarization of the scattered beam.

*N. Chako (New York)*

6351:

Slansky, Serge; Maréchal, André. Images en éclairage partiellement cohérent dans le cas de faibles contrastes. *C. R. Acad. Sci. Paris* 250 (1960), 4132-4134.

In this note the authors consider the problem of expressing the intensity produced by a partially coherent illuminated object having low contrast at a point in the image space. They derive two formal expressions of convolution type for the intensity which are in some respects simpler than the general formula. One of these is equivalent to that of a coherent object where  $E(M')$  representing the complex amplitude distribution over the pupil plane is replaced by  $EE_c$  which characterizes the image of a small isolated area (detail of the object) at the image point. The second formula gives the image intensity due to an incoherent object by taking half of the expression  $E_cE^* + E_c^*E$  as the image of a point, plus a contribution which represents a variation in phase in the object by taking  $E_cE^* - E_c^*E$  as the image of a dephased point. The quantity  $E_c$  corresponds to the complex amplitude distribution at a diffraction point from a small spot of the object after passing through the pupil or condenser of the system. The results can be extended to include aberration effects on the image quality.

*N. Chako (New York)*

6352:

Gordon, E. I. Transverse electron beam waves in varying magnetic fields. *Bell System Tech. J.* 39 (1960), 1603-1616.

The properties are studied of electron cyclotron and synchronous waves in magnetic fields which vary in time and space. The problem is treated by establishing the wave excitation from the knowledge of the macroscopic beam motion. Conclusions are drawn as to the phase velocity and the kinetic power carried by the waves.

*J. E. Rosenthal (Passaic, N.J.)*

6353:

Gautier, P. Contribution à l'étude des champs magnétiques de l'optique électronique. *Ann. Fac. Sci. Univ. Toulouse* (4) 21 (1957), 89-183 (1959).

The first part of this paper is a theoretical study concerning the possibility of measuring magnetic fields by coils of finite dimensions. It is shown that, to the fourth order, it is always possible to have the flux of magnetic induction through the coil proportional to the value of  $B$  at a single point. This consequence of Laplace's equation  $\Delta B = 0$  fixes the design of the coil. The second and third parts of the paper discuss the experimental arrangements and experimental data.

*J. E. Rosenthal (Passaic, N.J.)*

6354:

Sinel'nikov, K. D.; Rutkevich, B. N.; Fedorchenko, V. D. The motion of charged particles in a spatially periodic magnetic field. *Ž. Tehn. Fiz.* 30 (1960), 240-255 (Russian); translated as *Soviet Physics. Tech. Phys.* 5, 229-235.

The motion of a single charged particle in a magnetic field with small periodic variations along the direction of the field is described by a set of non-linear equations. The equations are solved by a perturbation technique, and the possibility is indicated of containing the particle in a magnetic trap for a considerable length of time by an appropriate choice of velocity and spatial periodicity.

*E. T. Kornhauser (Providence, R.I.)*

6355:

Grinberg, G. A.; Šukeilo, I. A. Method of solving a class of axially symmetric problems in the theory of potential and application to the design of electron-optical lenses. *Z. Tehn. Fiz.* **29** (1959), 1293-1303 (Russian); translated as Soviet Physics. Tech. Phys. **4** (1960), 1189-1198.

The aim of the authors is to obtain approximate solutions of a certain class of three-dimensional potential problems which would be useful in the practical design of electron optical systems. Their analysis is limited to systems of axial symmetry. By representing the curve  $C$  which is formed by the intersection of the conductor with the semi-plane  $\theta=0$  in parametric form, the authors have derived an integral representation of the potential at any point in space  $(r, z)$  involving an elliptic integral, and the charge distribution over the surface of the conductor. By combining this potential with the external potential field and equating this sum to the potential on the surface of the conductor, an integral equation is obtained for the charge distribution. Approximate solutions are given for the case points of the curve  $C$  lie closer to the axis of symmetry than those of the conductor. The special problems treated here are: (a) free distribution of charge on the surface of a circular torus; (b) on the surface of a single cylindrical lens formed by rotating a line segment about the  $z$ -axis, which is parallel to it; (c) a cylindrical lens with an internal aperture (a form  $T$  rotated about the  $z$ -axis). The approximate values of the capacity for problem (a) are compared to the exact values derived by means of toroidal coordinates. The difference between the two sets of values is rather small and, even for the value of the ratio of the radius of the circle, which by rotation about the  $z$ -axis forms the ring, to the radius of the cross section of the ring itself, equal to one half, this difference is less than 7% of the exact value. Approximate expressions for the charge distribution on these surfaces are obtained in terms of the potential on the conductor and the quantities describing the geometry of the conductor. *N. Chako (New York)*

6356:

Berger, J. M.; Bernstein, I. B.; Frieman, E. A.; Kulsrud, R. M. On the ionization and ohmic heating of a helium plasma. *Phys. Fluids* **1** (1958), 297-300.

The authors begin by stating the problem of heating the plasma to thermonuclear temperatures in a stelerator. They then analyze qualitatively the sequence of events following the application of a steady electric field around the stelerator (supplied by a steadily rising magnetic field). They then proceed to write down the conservation equations for energy and particle conservation. (As pointed out by R. Gerwin, their equation (10) for conservation of nuclei is misprinted;  $n_+$  should replace  $m_+$  in it.) In this they make use of an interpolation in the Spitzer (and Härm) calculation of the resistivity of a fully ionized gas. The authors point out that the Spitzer and Härm formula makes use of the assumption that the electron distribution is nearly Maxwellian. Thus, the electric field must be assumed so small that the critical electron energy (above which the electrons are continually accelerated rather than approaching a terminal velocity) is much larger than the mean thermal energy in the assumed Maxwellian. The energy dependent cross sections

for the various excitation and ionization processes in helium were approximated by analytic fits to experimental results. The equations were solved on a Univac for the case of a field strength of 0.11 volts/cm., a concentration of  $2.4 \times 10^{13}$  helium nuclei per  $\text{cm}^3$ , and effective inductance of 5.6 microhenries and a cross sectional area of 10 cm. The results are presented in graphs agreeing with the earlier qualitative arguments and showing limitation to the ion heating due to the small electron-ion momentum transfer cross section and the effects of Bremstrahlung. Unfortunately, the critical field is quite comparable to the electron temperatures developed so the conductivity approximation used is not valid. However, the authors assert (with reference to experimental studies) that the results are qualitatively correct.

*J. E. Drummond (Seattle, Wash.)*

6357:

Berger, J. M.; Newcomb, W. A.; Dawson, J. M.; Frieman, E. A.; Kulsrud, R. M.; Lenard, A. Heating of a confined plasma by oscillating electromagnetic fields. *Phys. Fluids* **1** (1958), 301-307.

The authors draw attention to the problem of heating the plasma to thermonuclear temperatures in a stelerator. They consider in this paper four cases of alternating electric fields normal to the main confining magnetic field. These (six possible) cases are defined by the direction a  $\ll$  sign between two of the important naturally occurring periods,  $\tau_{\text{coll}}$  (the mean free time between collisions of an ion) and  $\tau_{\text{tr}}$  (the mean transit time of an ion through the heating region) and an approximate equality between the period of the heating field and  $\tau_{\text{coll}}$  or  $\tau_{\text{tr}}$  or the ion cyclotron period,  $\tau_{\text{cy}}$  (which was always taken to be much less than either  $\tau_{\text{coll}}$  or  $\tau_{\text{tr}}$ ). After making a thermodynamic argument that heating rather than cooling can be expected, they write down and solve the simple approximate dynamical equations that apply to each of the four cases considered. They then show in one (and imply for the other) of the two cases when waves are directly produced by the excitation, that many different wave lengths are possible and thus that fine scale phase mixing can be expected to effectively thermalize the energy rapidly. (As R. Hall has pointed out, they omitted one term in their equation (29) for the perturbed current density; namely that arising from the density perturbation of the stream of velocity,  $V_1$ . However, their qualitative conclusion is still believable.) Finally, they compare and contrast the merits of the four cases studied.

*J. E. Drummond (Seattle, Wash.)*

6358:

Backus, George. Linearized plasma oscillations in arbitrary electron velocity distributions. *J. Mathematical Phys.* **1** (1960), 178-191; erratum, 559.

Mathematical investigation of the existence, uniqueness and stability of the linearized Vlasov equation. First it is shown that no plane wave solution can grow faster than  $e^{\omega_p t}$  ( $\omega_p$  is the plasma frequency), so that Landau's method of Laplace transformation covers all solutions. The remaining discussion is based on the familiar function

$$\mathcal{L}(s) = \frac{k^2}{\omega_p^2} - \int \frac{f_0(v)dv}{(s - k \cdot v/k)^2}$$

where  $k$  is the wave vector of the plane wave, and  $f_0(v)$  the equilibrium distribution. If  $\mathcal{L}(s)$  has zeros in the

upper half plane, there are exponentially increasing solutions connected with them. If  $\mathcal{L}(s)$  has no zeros in the upper half plane or on the real axis, there may still be unstable solutions, but they can be excluded by imposing square integrability on the initial disturbance.

The method of normal modes is discussed and extended to the case where  $\mathcal{L}(s)$  has zeros in the upper half plane [see also K. M. Case, *Ann. Physics* 7 (1959), 349-364; MR 21 #4741]. Finally, it is pointed out that the neglected nonlinear term grows linearly with time and may therefore vitiate stability considerations based on the linear theory.

{The author's discussion of the method of normal modes is somewhat misleading. For a non-vanishing, spherically symmetrical  $f_0(v)$  it yields all solutions, and they turn out to be stable. Hence no additional restrictions to assure stability are required.}

N. G. van Kampen (Utrecht)

6359:

Weibel, Erich S. Oscillations of a nonuniform plasma. *Phys. Fluids* 3 (1960), 399-407.

The author begins by recalling the state of a plasma confined by radiation pressure and then epitomizes properties of such states by means of a simple idealized mathematical model. It is the frequencies of longitudinal vibrational perturbations of the idealized state that are the objects of the analysis. He obtains a formal solution of the one dimensional perturbed Boltzmann equation by the method of characteristics and uses this with the one dimensional Poisson equation to give a homogeneous linear integro-differential equation for the perturbing electric field strength. The time variable can be eliminated since the differential and integral operators commute with  $\partial^2/\partial t^2$ , showing that these two operators have common eigenfunctions. The author chooses  $e^{-i\omega t}$  which then drops out of the equations leaving an integro-differential equation in one variable (position  $X$ ). In the succeeding section, he expands the unknown in a series of Hermite polynomials and by a succeeding integration by parts, use of recursion relations and a previously unpublished integral theorem for Hermite polynomials [which he proves in an appendix], he arrives at an equation of the form

$$(A) \quad \sum_{n=0}^{\infty} a_n H_n(kx) = e^{-(kx)^2} \sum_{m=0}^{\infty} b_m H_m(kx),$$

where  $b_m$  is proportional to  $a_{m-2}$  times a function of  $m$ . In a second appendix he derives the transformation matrix  $R_{n,m}$ :

$$(B) \quad a_n = 2 \sum_{m=0}^{\infty} R_{n,m} b_m$$

and then, using the dependence of  $b_m$  on  $a_{m-2}$ , finds the infinite linear homogeneous set of algebraic equations for  $a_n$ . The frequencies are to be determined by the requirement that the secular determinant be zero. To approximate this condition, he sets principal sub-determinants equal to zero but finds that the process does not converge with increasing size of the sub-determinant. However, by transforming the dependent variable in the infinite set of equations to the  $b_m$ 's rather than the  $a_n$ 's, he finds that the process of setting the successive sub-determinants of the secular determinant of the  $b_m$  equations equal to zero does converge. This is due to the better convergence of right hand side of (A) above.

In the next section, he obtains an analytic approximation to the eigenfrequencies in the limit of very low plasma density. At zero density the characteristic frequencies are integral multiples of the natural frequency of the confinement well and are, as he also shows, infinitely degenerate. Then, as the density is allowed to increase from zero, he finds that the degeneracy is removed and many different frequencies grow from each of the original ones. One such result is:  $\omega^2 = \omega_0^2 + 0.731\omega_p^2$ ,  $\omega_p^2 = 4\pi ne^2/m$ ,  $\omega_0$  is the natural frequency of the well. He supports this by numerical calculations for densities up to where  $\omega_p^2 = 16\omega_0^2$ .

In his final section he derives the results on the basis of a hydrodynamic rather than a Vlasov model; showing that hydrodynamics leads to an incorrect spectral change with density.

J. E. Drummond (Seattle, Wash.)

6360:

De Socio, Marialuisa. Su fronte di un'onda elettromagnetica in un gas ionizzato soggetto a un campo magnetico. *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat.* (8) 27 (1959), 368-373.

The author studies the behavior of the fields  $E$  and  $H$  on the wave front of a plane (non-sinusoidal) electromagnetic wave propagated along the  $z$ -axis in an ionized medium (gas) subjected to a constant magnetic field  $H_0$ , normal to the  $z$ -axis. The results are based on the following assumptions.  $E$  and  $H$  are continuous for  $z \geq 0$  and  $t \geq 0$  and vanish for  $t < 0$ . For  $z = 0$ , their first time derivatives are discontinuous, and furthermore in the Maxwell equations only the electrons contribute to the current term (convection), which is determined from Lorentz equation of motion, satisfying the boundary condition  $v_x = 0$  for  $z = 0$ . By resolving the electromagnetic wave at the plane  $z = 0$  in two waves, one parallel to the  $x$ -axis (transverse) and the other parallel to the  $y$ -axis, the author has derived the following results: The fields and their first time derivatives (partial) remain constant over the wave front as the transverse wave propagates in the ionized gas. On the other hand, the total derivatives with respect to time  $t$  of the second partial time derivatives do not vanish on the wave front, showing that the second partial time derivatives of the field do not vanish on the wave front, except at  $z = 0$ . Similar deductions are obtained for the second wave, which contains a longitudinal component of the third order. These results are valid provided the Love condition is satisfied on the wave front. If the medium is non-homogeneous, i.e., if  $N$ , the number of the electrons, and the collision parameter (number) depend on position ( $z$ ), the above properties of the fields on the wave front are still satisfied.

N. Chako (New York)

6361:

Platzman, P. M.; Osaki, H. T. Scattering of electromagnetic waves from an infinitely long magnetized cylindrical plasma. *J. Appl. Phys.* 31 (1960), 1597-1601.

The authors determine by the Fourier-Lamé method the differential scattering cross-section of a homogeneous plasma cylinder biased by a longitudinal magnetostatic field. When the normally incident wave is polarized such that its electric vector is parallel to the axis of the cylinder, the problem reduces to the trivial one of finding the cross-section of an homogeneous isotropic dielectric cylinder.

However, when the incident wave is polarized such that its electric vector is perpendicular to the axis of the cylinder, the anisotropy of the plasma plays a major role. For this case they find an analytical expression for the differential scattering cross-section, taking into account the collisions and the finiteness of the biasing field, and obtain numerical results which place in evidence the various types of resonance that can occur.

C. H. Papas (Pasadena, Calif.)

6362:

Montgomery, David. Stability of large amplitude waves in the one-dimensional plasma. *Phys. Fluids* 3 (1960), 274-277.

An investigation, based on the Vlasov equation, of the stability of a special class of non-linear traveling waves in a plasma. The class of waves is restricted to those which are composed of particles whose velocities are all in the same direction. Because of this restriction the eigenvalue equation for the perturbed distribution function can be formally solved in powers of the ratio of the maximum potential energy to the minimum particle energy. An argument is then given which purports to show that a non-linear wave is stable if the corresponding lowest order distribution function is stable.

E. A. Jackson (Princeton, N.J.)

6363:

Stix, Thomas H. Absorption of plasma waves. *Phys. Fluids* 3 (1960), 19-32.

A study of the absorption and reflection of waves in a plasma, wherein the density and/or magnetic field are slowly varying functions of a single spatial coordinate,  $x$ . A number of cases are discussed where the dispersion relations for a cold, collisionless plasma are of the form  $bk^2 + c = 0$  ( $k$  is the propagation constant in the  $x$  direction). Treating  $b$  and  $c$  as slowly varying functions of  $x$ , the WKB (adiabatic) approximation is applied in those regions separated by a zero or singularity of  $c/b$ . The wave functions in the two regions are joined by the usual linear connection method, and it is shown that this predicts total reflection [absorption] if  $c/b$  has a simple zero [pole]. These results are re-examined using corrected dispersion relations,  $ak^4 + (b_r + ib_i)k^2 + c = 0$ , which take into account thermal and collisional effects and higher order corrections in the electron to ion mass.

E. A. Jackson (Princeton, N.J.)

6364:

Spitzer, Lyman, Jr. The stellarator concept. *Phys. Fluids* 1 (1958), 253-264.

Author's summary: "The basic concepts of the controlled thermonuclear program at Project Matterhorn, Princeton University, are discussed. In particular, the theory of confinement of a fully ionized gas in the magnetic configuration of the stellarator is given, the theories of heating are outlined, and the bearing of observational results on these theories is described.

"Magnetic confinement in the stellarator is based on a strong magnetic field produced by solenoidal coils encircling a toroidal tube. The configuration is characterized by a 'rotational transform', such that a single line of magnetic force, followed around the system, intersects a cross-sectional plane in points which successively rotate about the magnetic axis. A theorem by Kruskal is used to

prove that each line of force in such a system generates a toroidal surface; ideally the wall is such a surface. A rotational transform may be generated either by a solenoidal field in a twisted, or figure-eight shaped, tube, or by the use of an additional transverse multipolar helical field, with helical symmetry.

"Plasma confinement in a stellarator is analyzed from both the macroscopic and the microscopic points of view. The macroscopic equations, derived with certain simplifying assumptions, are used to show the existence of an equilibrium situation, and to discuss the limitations on material pressure in these solutions. The single-particle, or microscopic, picture shows that particles moving along the lines of force remain inside the stellarator tube to the same approximation as do the lines of force. Other particles are presumably confined by the action of the radial electric field that may be anticipated.

"Theory predicts and observation confirms that initial breakdown, complete ionization, and heating of a hydrogen or helium gas to about  $10^4$  degrees K are possible by means of a current parallel to the magnetic field (ohmic heating). Flow of impurities from the tube walls into the heated gas, during the discharge, may be sharply reduced by use of an ultra-high vacuum system; some improvement is also obtained with a divertor, which diverts the outer shell of magnetic flux away from the discharge. Experiments with ohmic heating verify the presence of a hydromagnetic instability predicted by Kruskal for plasma currents greater than a certain critical value and also indicate the presence of other cooperative phenomena. Heating to very much higher temperatures can be achieved by use of a pulsating magnetic field. Heating at the positive-ion cyclotron resonance frequency has been proposed theoretically and confirmed observationally by Stix. In addition, an appreciable energy input to the positive ions should be possible, in principle, if the pulsation period is near the time between ion-ion collisions or the time required for a positive ion to pass through the heating section (magnetic pumping)."

M. Kruskal (Princeton, N.J.)

6365:

Gurevič, A. V. On the amount of accelerated particles in an ionized gas under various accelerating mechanisms. *Ž. Eksper. Teoret. Fiz.* 38 (1960), 1597-1607 (Russian. English summary); translated as Soviet Physics. JETP 11, 1150-1157.

Attention is first drawn to some of the semi-quantitative features of the statistical acceleration of ions in the sun (or perhaps elsewhere) which could give rise to the high energies observed for cosmic ray particles. Some of the pitfalls in these considerations are pointed out. Next, a dimensionless Fokker-Planck type equation including a Fermi acceleration term (a model of continual reflection from moving magnetic fields) but excluding ion-electron interaction terms is written down from the literature. Familiarity with this type of equation is gradually developed; first by obtaining the time independent solution of the equation, then by an iteration procedure together with a separation of the velocity space into regions where different approximations apply, the author develops a quantitative result for the rate of acceleration of particles into the "run-away" region. (This is the region of energies greater than a certain energy he calls the "injection energy",  $\varepsilon_{in}$ , but which some Western authors

call the critical energy. The collisional drag of other charged particles on an ion decreases asymptotically so rapidly with energy that continual acceleration rather than an approach to a terminal velocity occurs.) Finally, he includes the drag due to electrons (which is much more important than that due to other ions) and a more general acceleration law. For this class of cases, he obtains two limiting solutions involving two functions defined by integrals. He tabulates them at five points and discusses their nature and significance relative to the acceleration problem.  
J. E. Drummond (Seattle, Wash.)

6366:

Bickerton, R. J. Coulomb collisions and plasma conductivity. *Rend. Scuola Internaz. Fis. "Enrico Fermi"*, Corso XIII (1959), pp. 107-118. Zanichelli, Bologna, 1960.

6367:

★Fisica del plasma: Esperimenti e tecniche. *Rendiconti della Scuola Internazionale di Fisica "Enrico Fermi"*. Corso XIII, 2-15 settembre 1959 (Direttore: H. Alfvén). Nicola Zanichelli, Bologna, 1960. xii + 166 pp. (1 plate). L. 2000.

A set of twenty lectures on plasma physics. Fifteen (those of mathematical character) will be given separate reviews.

6368:

Jackson, E. Atlee. Drift instabilities in a Maxwellian plasma. *Phys. Fluids* 3 (1960), 786-792.

The dispersion relation for electrostatic waves in a plasma consisting of two Maxwellian components is analyzed by a graphical method. Regarding the thermal speeds and Debye lengths of the two components as fixed, the method yields, as a function of instability wavelength, the two values for the relative drift velocity such that instabilities can grow at a given rate. This paper concentrates mainly on the case of zero growth rate, and the values considered for the ratio of the Debye lengths are 1, 4, 10, and very large values. O. Penrose (London)

6369:

Kippenhahn, R.; de Vries, H. L. Zur Energieabgabe eines modulierten Ionenstrahls im Plasma mit Magnetfeld. *Z. Naturforsch.* 15a (1960), 506-512. (English summary)

This paper contains a study of the effect of an ion stream of velocity  $v$  on a plasma which is under the influence of a homogeneous magnetic field normal to the ion motion. The ion beam is confined to a narrow slit-like beam and is modulated with a frequency  $\omega$ . The results obtained here are based on the solution of Maxwell equations, without considering the motion of the ions in the plasma, thus restricting the solution to the case of a straight line motion of the ions in the plasma. The author distinguishes two situations, the field lines being: (a) parallel to the plane of the ion stream; (b) normal to this plane.

By assuming  $\omega$  to be small in comparison to the gyro-frequency,  $\omega_g$ , of the electrons and to the plasma frequency  $\omega_p$ , and furthermore if the plasma pressure is

neglected and of infinite conductivity, the mean relative energy loss of an ion per unit path is given by

$$(1) \quad 2\pi r \frac{N_2^2}{N_1} \frac{Q+P-1}{P(P-1)^{1/2}}, \quad P > 1, \quad (r = e^2/mc^2),$$

where  $N_1$ ,  $N_2$  and  $r$  denote the number of particles per unit area of the un-modulated and modulated beam and the ion radius respectively. The quantities  $Q$  and  $P$  are  $(\omega/\omega_g)^2$  and  $(v/v_a)^2$ ,  $v_a$  being the Alfvén velocity. For low density of the ions the loss is insignificant unless the path of the ions in the plasma is very large (not realized in practice). However, for  $P=1$ , resonance takes place between the ions and the plasma, therefore an appreciable amount of energy is transferred from the ions to the plasma, provided the conductivity is finite. The resonance case, however, is not treated in this paper.

N. Chako (New York)

6370:

Colgate, Stirling A. Collisionless plasma shock. *Phys. Fluids* 2 (1959), 485-493.

Author's summary: "The structure of an extremely strong magnetohydrodynamic shock is discussed in the limit of no particle collisions. It is tentatively concluded that the shock transition takes place through the mechanism of a strong electric field produced by charge separation. The pressure in the shocked plasma is due primarily to a very high electron temperature. The entropy increase occurs by Landau damping of the coherent electron oscillation. The ions, on the other hand, undergo an irreversible temperature change of only 3."

H. Cabannes (Paris)

6371:

Solymar, L. Stability of a plasma sheet in time periodic magnetic fields. *J. Electronics Control* (1) 9 (1960), 391-396.

Author's summary: "The behaviour of a thin sheet of electrons in a time-varying square wave magnetic field is investigated. It is shown that the growth of an initial disturbance which takes place in the first half of a period is entirely annulled in the second half, both the displacements and velocities returning to their initial values. It is concluded from this that the break up of annular beams in a uniform magnetic field which has been observed should be strongly inhibited in a periodic field."

6372:

Franklin, R. N. Space-charge neutralization and thermionic emission. *J. Electronics Control* (1) 9 (1960), 385-390.

Author's summary: "A hot surface emitting charged particles of both signs is placed near a non-emitting collector maintained at a potential relative to the emitter. The potential in the region between the electrodes is investigated. An exact solution of the density distribution of space charge and the field is derived. Though no analytical expression can be given for the potential distribution, its general shape can be found as a function of the ratio of the number of positive ions to electrons. It is shown that the transition from pure electron emission to ion emission proceeds smoothly and no oscillatory solution for the potential exists."

6373:

Johnston, T. W. Time-averaged effects on charged particles in a-c fields. *RCA Rev.* **21** (1960), 570-610.

Author's summary: "When a nonuniform a-c field is applied to a charged particle it experiences a time-averaged acceleration in addition to the a-c acceleration. The development and application of the basic theory to plasmas is outlined, including some cases with a d-c magnetic field. A fairly complete bibliography, and a critique of theory and experiments are also given."

6374:

Klimontovič, Yu. I. A relativistic transport equation for a plasma. II. *Ž. Eksper. Teoret. Fiz.* **38** (1960), 1212-1221 (Russian. English summary); translated as *Soviet Physics. JETP* **11**, 876-882.

This paper is the continuation of an earlier paper by the same author in same *Ž.* **37** (1959), 735-744 [MR **22** #3168]. A relativistic transport equation for a plasma is derived, neglecting radiation. After that a relativistic Fokker-Planck equation is derived in which diffusion and friction arising from the excitation of plasma oscillations is taken into account. *D. ter Haar* (Oxford)

6375:

Silin, V. P. On the electromagnetic properties of a relativistic plasma. *Ž. Eksper. Teoret. Fiz.* **38** (1960), 1577-1583 (Russian. English summary); translated as *Soviet Physics. JETP* **11**, 1136-1140.

The Vlasov equations which neglect collisions are used to derive an expression for the dielectric constant, the screening constant, and the skin-depth. The plasma oscillations were assumed to be either not damped, or very slightly damped. *D. ter Haar* (Oxford)

6376:

Harris, Gilda Maki; Roberts, John E.; Trulio, John G. Equilibrium properties of a partially ionized plasma. *Phys. Rev.* (2) **119** (1960), 1832-1841.

Authors' summary: "A model for a partially ionized, partially dissociated plasma has been formulated using known theoretical concepts to describe both bound and free electron states, internal molecular degrees of freedom and Coulomb interactions. It has been applied to a system of particles arising from the hydrogen molecule. The Coulomb interaction is treated in the classical Debye approximation. However, a distance of closest approach between ions and electrons depending on the kinetic energy of the electrons is included to avoid the short-range divergence of the Coulomb potential. The kinetic energy of the free electrons is calculated from the partition function for a perfect Fermi gas. The vibrational and rotational motion are treated in the harmonic oscillator and rigid rotor approximation with the number of energy levels counted for a given electronic state depending on the dissociation energy of the state. A volume dependence of the bound electronic energy eigenvalues is included by considering the effect of surrounding particles as a confinement of a given particle to a spherical box of variable size. For the counting of the bound electronic states, a given state is bound until its energy increases to zero due to confinement. From the partition function for the

entire system, the free energy is calculated. By a minimization of the free energy of the system, the equilibrium composition as a function of temperature and volume is obtained. Then not only can thermodynamic quantities be calculated, but it is believed that a reasonable approximation to the correct balance of molecular, ionic and free electronic states is achieved over a wide range of  $v$ - $T$  space. Consequently, regions where incomplete ionization and dissociation are important are delineated. In addition, for different regions of  $v$ - $T$  space, the relative contributions of charged particle interaction of the nonclassical behavior of electrons, of internal degrees of freedom and of translation to the total energy of the system can be determined."

6377:

Landshoff, Rolf K. M. (Editor). ★The plasma in a magnetic field: A symposium on magnetohydrodynamics. Stanford University Press, Stanford, Calif., 1958. vii + 130 pp. \$4.50.

This book is the second in a series of symposia on magnetohydrodynamics edited by Dr. Landshoff. The volume comprises a collection of ten short articles by as many authors on a number of topics. It has been divided by the editor into three sections: (1) kinetic theory; (2) confinement and instabilities of a plasma; and (3) high-speed fluid dynamics. The subject matter is more various than this grouping indicates, however, and the book cannot easily be characterized in a general way except to say that no part of it depends upon any other. Indeed, it appears to be simply a representative sample from the proceedings of a scientific meeting. With the exception of two introductory articles, which are in the nature of brief, formal reviews of some general theoretical principles, the contributions are not noticeably different from those one might find in a typical issue of, say, the "Physics of Fluids".

There are three theoretical papers in addition to those in the introductory section. One of these deals with the linearized theory of the interaction between a plasma column and the electromagnetic field in a concentric conducting cylinder which is excited in the symmetric  $E$ -mode. Another is a rather general discussion of the stability of a tenuous plasma in the interplanetary magnetic field. The third presents detailed results of the calculation of flow characteristics for a conducting fluid in an infinite channel whose parallel walls are in relative motion in a direction normal to a steady, uniform magnetic field.

Five of the articles are discussions of particular experimental studies of plasma under a variety of conditions of production and measurement. In one of these, pinch instabilities in a discharge tube with longitudinal magnetic fields up to 3,000 gauss are investigated using both external and internal magnetic probes. Photo- and oscillographic records reveal the expected pinch modes. Even more striking perhaps is a three-dimensional representation of the measured current density distribution function over a cross section which portrays the progress of the pinch phenomenon in time. Three of the remaining experimental papers deal with high-velocity plasmas produced by strong shocks in a low-pressure gas. These contain descriptions of the means of generating the shock, measurements of the shock velocity, and less direct

determinations of temperature, energy content, etc. In one of these papers it is demonstrated that a shock-induced plasma can be constricted by a strong magnetic field.

It has often been remarked that one of the major problems of plasma research has to do with the difficulty experienced by theoreticians in dealing with the physical situations encountered by experimenters, and the inability of experimenters to produce physically the extremely simple conditions preferred by many theoreticians. Except when they are discussing gross effects in a plasma, the two groups seem to proceed independently of each other. The record of this symposium is further evidence that the problem is still a serious one.

R. Kodis (Providence, R.I.)

6378:

Jeans, James. ★The mathematical theory of electricity and magnetism. 5th ed. Cambridge University Press, New York, 1960. viii + 652 pp. \$3.95.

Paperbound reprinting of the 5th [1925] edition.

6379:

Zin, Giovanni. Sui fondamenti dell'elettrodinamica. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 28 (1960), 183-188, 368-370.

These notes are concerned with the mechanical forces acting on a linear conductor carrying a current in the presence of a magnetic field of external origin. The results proved are as follows. (I) In a homogeneous medium of permeability  $\mu$  the action of an element  $ds_1$  of a conductor carrying current  $I_1$  and an element  $ds_2$  of a second conductor carrying current  $I_2$  consists of a force

$$d^2F = -\frac{\mu I_1 I_2}{4\pi} (ds_1 \cdot ds_2) \frac{r}{r^3}$$

and a couple

$$d^2M = -\frac{\mu I_1 I_2}{4\pi r} ds_1 \times ds_2$$

where  $r$  is the vector from this first element to the second. (II) The action of magnetic induction  $B$ , derived from a vector potential  $A$ , on the element  $ds$  of an inextensible conductor carrying current  $I$  consists of a force  $-B \times Ids + (dA/ds)Ids$  and a couple  $-A \times Ids$ . (III) If the conductor is extensible, there is also a mechanical tension  $A \cdot I(ds/ds)$ . (IV) Magnetic induction  $B$  derived from a vector potential  $A$  exercises on an arc  $(a, b)$  of a conductor  $C$  carrying a current  $I$  an elementary force  $-B \times Ids$  on each element  $ds$  of  $C$ , a force  $-IA$  at the end  $a$  and a force  $IA$  at the end  $b$ .

E. T. Copson (St. Andrews)

6380:

Toupin, R. A.; Rivlin, R. S. Linear functional electromagnetic constitutive relations and plane waves in a hemihedral isotropic material. Arch. Rational Mech. Anal. 6, 188-197 (1960).

In the original formulation of electromagnetic theory by Faraday and Maxwell the properties of material media were described by characteristic constants such as the dielectric constant and magnetic permeability. Much work has been done in the determination of these constants in terms of the molecular properties of matter. In another

line of thought, of particular appeal to mathematicians, the material constants are replaced by various formal relations among the electromagnetic field vectors. The authors investigate the consequences of relations of the general form

$$D(x, t) = \Phi[E(x, \tau)_{-\infty}^t, B(x, \tau)],$$

$$H(x, t) = \Psi[E(x, \tau)_{-\infty}^t, B(x, \tau)],$$

where the right hand sides are linear functionals depending on the values of the indicated field vectors over the time interval  $-\infty < \tau \leq t$ . In contrast with the usual physical theory, this assumption credits the medium with a long-range memory property. For the actual calculations of the paper these functionals are assumed to be of somewhat more specialized types.

The propagation of plane waves and the resulting dispersion relation are investigated for a medium of this character. The physical questions of energy density and energy flux are not considered.

E. L. Hill (Minneapolis, Minn.)

6381:

Sivkova, V. V. Calculation of the values of the vector potential of the field of a cylinder placed in a variable field of a solenoid. Tomsk. Gos. Univ. Uč. Zap. No. 25 (1955), 115-121. (Russian)

[This article was listed in MR 19, 1011.]

The author first calculates the vector potential due to a sinusoidal current passing through an infinite helix, the result appearing as an integral involving Bessel functions. An infinite cylindrical conducting core is then centered inside the helix and the vector potential of this configuration calculated, again as an integral in terms of Bessel functions. These results are further extended to the case of  $2n$  helices distributed symmetrically with axes parallel. The formula resembles that for the vector potential of a plate placed in the varying field due to a system of linear currents. A tabulated comparison of numerical values shows that, for the particular parameters selected, the discrepancy between the two situations is small.

M. G. Arsove (Seattle, Wash.)

6382:

Byuler, G. A. Cylindrical conductor of arbitrary cross-section in a plane periodic field. Tomsk. Gos. Univ. Uč. Zap. No. 25 (1955), 122-134. (Russian)

A plane periodic magnetic field is imposed perpendicularly on a cylindrical conductor whose cross-sectional boundary consists of a piecewise analytic curve. The longitudinal electric field is then expressed in terms of an integral equation resulting from an application of Green's formula to Maxwell's equations. A method of solution of the integral equation by successive approximations is presented, and first and second approximations are worked out for the case of circular cross sections.

M. G. Arsove (Seattle, Wash.)

6383:

Papapetrou, Achille. Une représentation du champ de radiation électromagnétique. C. R. Acad. Sci. Paris 250 (1960), 4292-4294.

Instead of the four functions (potentials) satisfying both the Maxwellian equations  $\square A^\mu = 0$  and the Lorentz

(divergence) condition  $\Delta \mu = 0$ , the author sketches a formal proof which shows that the same class of solutions can be obtained from two independent functions  $w$  satisfying  $\square w = 0$ .  
M. Kline (New York)

6384:

Papapetrou, Achille. Représentation d'un champ de radiation faible en relativité générale et dans la théorie d'Einstein-Schrödinger. C. R. Acad. Sci. Paris 251 (1960), 49-50.

Suite de #6383. L'auteur détermine ici le champ gravifique faible en relativité générale—il dépend de deux fonctions d'onde—et le champ électromagnétique en théorie d'Einstein-Schrödinger (système fort des équations) qui dépend de trois fonctions d'onde.

J. Renaudie (Montpellier)

6385a:

Green, H. S.; Wolf, E. A scalar representation of electromagnetic fields. I. Proc. Phys. Soc. Sect. A 66 (1953), 1129-1137.

6385b:

Wolf, E. A scalar representation of electromagnetic fields. II. Proc. Phys. Soc. 74 (1959), 269-280.

6385c:

Roman, P. A scalar representation of electromagnetic fields. III. Proc. Phys. Soc. 74 (1959), 281-289.

The mathematical description of optical phenomena developed during the first half of the 19th century by Fresnel, Kirchhoff, and others, made use of a single scalar field function. This "optical disturbance" function was supposed to be a solution of the wave equation, to be propagated according to Huygens' principle, and to determine the light flux and intensity. Although neither the physical nor the mathematical properties of this function were well-defined, it proved possible to use it to give a remarkably simple and accurate description of a wide class of interference and diffraction phenomena, with the exception of polarization. After the development of Maxwell's equations for the electromagnetic field, this system of vector equations gradually became the accepted basis of optical theory. Nevertheless, later developments have revealed a persistent nostalgia among optical theoreticians for the earlier theory, which is still widely used in texts and in the literature.

The papers under review present an attempt to find an exact formulation of the Fresnel-Kirchhoff scalar theory within the context of Maxwell's vector equations. The basis of the method is the expression of the vector field functions of a radiation field in terms of a complex-valued scalar function (complex potential). The mathematical techniques employed clearly owe some of their inspiration to procedures currently in vogue in quantized field theories.

I. In the first paper the authors formulate the definition of the complex potential of a radiation field. Using a gauge in which the usual scalar potential vanishes identically, the vector potential is expressed in terms of a Cauchy-Fourier integral over the component waves of the

field. A construction resembling the use of Stokes-Poincaré parameters is then employed to reformulate this vector integral in terms of a single complex-valued scalar integral which defines the complex potential. The complex-valuedness of the potential reflects the transverse polarization of the component waves of the radiation field.

The authors regard the energy density and energy flux in the radiation field as unphysical quantities, not determined uniquely by the field vectors. It is shown that definitions can be given for these quantities which satisfy a simple conservation law. A formal description of wave fronts and rays is found which is not restricted to the case of short wavelengths.

{A potential obscurity in the theory arises from a failure (customary in the physical literature) to distinguish between Fourier integral analysis and eigenfunction expansions. Ostensibly the theory is restricted to an  $L_2$ -framework, equivalent to the use of wave packets, but consistency on this point is not maintained in this, or in the following papers.}

II. The author gives first a more elegant formulation of the construction of the integral by which the complex potential is defined. This integral is then separated into two parts, called positive and negative partial waves, the one containing the time-variable member  $\exp(i\omega t)$ , and the other  $\exp(-i\omega t)$ , with  $\omega \geq 0$ . It is shown that the corresponding parts of the complex potential describe fields which are mutually incoherent, on a time average, and in a sense are propagated independently.

III. An attempt is made in this paper to enlarge the scope of the theory by bringing it into contact with more general field concepts. In particular, the definition of the energy density and energy flux is reconsidered in the light of the requirement of the general theory of relativity that these quantities are real physical entities and so must have definite (tensor) formulations. A special investigation becomes necessary since the present theory is invariant only under a combination of Lorentz and gauge transformations, owing to the initial choice of gauge in the definition of the radiation field. It is concluded that the definitions given in the present theory are compatible with the requirements of the general theory of relativity.

{The reviewer finds it difficult to place this work in the more general framework of present-day theoretical physics. The approach is distinctly formalistic in character, and appears to leave aside the whole molecular theory of the optical properties of matter which has become associated with the electromagnetic theory of optics. It remains to be seen how it is to be applied to boundary-value problems, in which the boundary conditions require physical interpretation.}

E. L. Hill (Minneapolis, Minn.)

6386:

Schlegel, Richard. Radiation pressure on a rapidly moving surface. Amer. J. Phys. 28 (1960), 687-694.

The net force on a stationary body due to radiation pressure in an isotropic electromagnetic radiation field is of course zero. When the body is moving at high velocities, the same radiation field appears anisotropic, and a net force results. It is shown, however, that this force remains finite as the velocity approaches the velocity of light and is of negligible magnitude for macroscopic bodies in a field having the energy density ascribed to interstellar space.  
E. T. Kornhauser (Providence, R.I.)

6387:

Maškevič, V. S. Electromagnetic waves in a medium with a continuous energy spectrum. I. *Ž. Eksper. Teoret. Fiz.* 38 (1960), 906-911 (Russian. English summary); translated as Soviet Physics. JETP 11, 653-656.

Suppose that the state vector of a system is expressed in terms of wave functions corresponding to a continuous portion of the spectrum. These wave functions are not localized in space, and as a consequence, when they are excited by an electromagnetic field, the polarization at any point of the medium will depend on the values of the electric field throughout the medium. In this case, the relation between the polarization vector and the electric field is no longer a point relationship, but is rather an integral relationship of the form

$$P(r, t) = \int K(r, r_0, \tau) \cdot E(r_0, t - \tau) dr_0 d\tau.$$

In this paper, the author determines the kernel  $K(r, r_0, \tau)$  for the case of a crystalline medium when anharmonic terms in the vibration potential are taken into account. He uses a variant of the Weisskopf-Wigner method, developed by Born and Huang, to construct the approximate state vector of the system. With the aid of this state vector, and by treating the electric field as a solenoidal classical field, the author derives the desired integral relationship. The Fourier transform of the kernel,  $K(r, r_0, \omega)$  is then split into its real and imaginary parts, which are shown to satisfy standard dispersion relations in  $\omega$ . These can be considered as a generalization of the Kramers-Kronig formulas. Finally, the case of a general, non-solenoidal electric field is considered.

J. McKenna (Murray Hill, N.J.)

6388:

Gersdorff, R. Uniform and non-uniform form effect in magnetostriction. *Physica* 26 (1960), 553-574.

Author's summary: "A calculation is made of the form effect observed in magnetostriction. On the assumption that the strains are uniform, a calculation of this effect is made for ellipsoids of rotational symmetry magnetized along a principal axis. This is done for elastically isotropic material and for an elastically anisotropic cubic crystal, with the cube axes in an arbitrary direction.

"A more accurate investigation showed, however, that in most cases the strains are far from uniform. An exact calculation of the strains as a function of the position could only be made for a sphere, an isotropic sphere and a monocrystalline sphere of cubic crystal symmetry, the magnetization being in a (110) plane. An approximate calculation is made for a few other cases that may be of importance for experiments in which magnetostriction constants are determined."

6389:

Lebedev, N. N.; Skal'skaya, I. P. Electrostatic field of an electron lens consisting of two coaxial cylinders. *Ž. Tehn. Fiz.* 30 (1960), 472-479 (Russian); translated as Soviet Physics. Tech. Phys. 5, 443-450.

The field distribution in a lens formed by two coaxial cylinders is determined exactly from the solution of paired integral equations. The formulas are used to

calculate the potential distribution along the axis of the lens for various values of the ratio of the radii of the inner and outer cylinders. *J. E. Rosenthal* (Passaic, N.J.)

6390:

Lindsay, P. A.; Parker, F. W. Potential distribution between two plane emitting electrodes. II. Thermionic engines. *J. Electronics Control* (1) 9 (1960), 81-111.

Authors' summary: "This paper gives expressions for the potential distribution between two plane parallel emitting electrodes, extending the work of a previous paper [Lindsay and Parker, same J. (1) 7 (1959), 289-315; MR 21 #7717]. It is shown that all potential distributions can be represented by a two-parameter family of curves, the parameters being the ratio of the electrode temperatures  $\theta = T_2/T_1$  and a constant  $A$  which depends on  $\theta$  and on the potentials and work functions of the two electrodes ( $A$  reduces to the parameter of Lindsay and Parker [op. cit.] for  $\theta = 1$ ). The results show rather clearly the relative influence of all these quantities on the position and depth of the potential minimum between the electrodes."

6391:

Ginzburg, V. L. Einige Fragen der Strahlungstheorie bei Bewegung mit Überlichtgeschwindigkeit in einem Medium. *Fortschr. Physik* 8 (1960), 295-326.

The present article is a translation from Russian into German. (The English translation appeared in Soviet Physics. Uspekhi 2 (1960), 874-893; see #6392.) The author subdivides his discussion into five sections: (1) Characteristic features of radiation due to superlight motion (Classical theory); (2) Quantum theory of radiation and absorption due to superlight motion; (3) Radiation reaction force for motion of a charge in a medium; (4) Cerenkov radiation and absorption of waves in an isotropic magnetoactive plasma; (5) Cerenkov radiation of dipole moments in a continuous medium and in slits and channels. This is an excellent review on the subject. The presentation is clear and complete. Perhaps of special interest are the following discussions: Section 2 contains an instructive analysis of the anomalous Doppler effect for radiation inside the Cerenkov cone; section 4 discusses the analogy between Cerenkov radiation and the emission of plasma waves by a charge moving with suprathermal velocities in a plasma; section 5 clears up the confusion which was prevalent in the literature concerning the Cerenkov radiation emitted by magnetic dipoles. The literature references are exhaustive.

N. L. Balazs (Princeton, N.J.)

6392:

Ginzburg, V. L. Certain theoretical aspects of radiation due to superluminal motion in a medium. *Uspekhi Fiz. Nauk* 69 (1959), 537-564 (Russian); translated as Soviet Physics. Uspekhi 2 (1960), 874-893.

German translation appears in *Fortschr. Physik* 8 (1960), 295-326 [see preceding review].

6393:

Hora, H. Zur Seitenversetzung bei der Totalreflexion von Materiewellen. *Optik* 17 (1960), 409-415. (English and French summaries)

Author's summary: "The transverse shift of a bundle of waves reflected in the region of total reflection, which is well known for electromagnetic and acoustic waves (Goos-Hänchen effect), is calculated for waves of matter. The Schrödinger wave equation enables a formulation to be made strictly in terms of a bundle of waves, whose width increases more rapidly than the magnitude of the shift of the rays as the critical angle of total reflection is approached. Furthermore, it emerges from the theory that a small difference in density between the adjoining media results in a fairly large shift, and it is pointed out that there is an analogous shift associated with the reflection of electromagnetic waves by the ionosphere."

6394:

Pisareva, V. V. Limits of applicability of the method of "smooth" perturbations in the problem of radiation propagation through a medium containing inhomogeneities. *Akust. Zh.* **6** (1960), 87-91 (Russian); translated as *Soviet Physics. Acoust.* **6**, 81-86.

In an inhomogeneous medium the time reduced scalar wave equation is  $(\nabla^2 + K_0^2 n^2(r))u = 0$ . In case the index of refraction,  $n(r)$ , is subjected to a small perturbation,  $n(r) = 1 + \mu(r)$  with  $|\mu(r)| \ll 1$ , the standard method of small perturbations is to develop  $u$  in powers of  $|\mu|$  and to solve the associated linearized perturbation equations. The method of 'smooth' perturbations developed by Obuhov [*Izv. Akad. Nauk SSSR. Ser. Geofiz.* **1953**, 155-165; *MR* **15**, 1002] consists in writing

$$u(r) = A_0 \exp[i(\psi_0 + \psi')].$$

Here  $v_0 = A_0 e^{i\psi_0}$  is the solution of  $(\nabla^2 + K_0^2)v_0 = 0$ . It is then found that  $\psi'$  satisfies the equation

$$2\nabla\psi_0 \cdot \nabla\psi' - i\nabla^2\psi' = 2\mu K_0^2 + \mu^2 K_0^2 - (\nabla\psi')^2.$$

The assertion is then made that the terms  $\mu^2 K_0^2$  and  $(\nabla\psi')^2$  may be neglected if  $|\nabla\psi'|/k_0 \ll 1$ . Thus this perturbation method is asserted to be valid if the change in phase and the relative change in amplitude per wave length are small, while the total change in these quantities is not restricted. The author studies the linear approximation in more detail. The author concludes that the applicability depends upon the size of the wave parameter  $D = 4L/l^2 K_0$ . Here  $L$  is the distance and  $l$  is the mean inhomogeneity scale. The author concludes that for  $D$  small the method is applicable if in addition the amplitude modulation is small, while, for  $D$  large the amplitude and phase fluctuations must be small. *B. Levy* (New York)

6395:

Karp, S. N.; Karal, F. C., Jr. Vertex excited surface waves on one face of a right angled wedge. *Quart. Appl. Math.* **18** (1960/61), 235-243.

The radiation of a magnetic line source on the vertex of a right angled wedge is calculated for a perfectly conducting surface on one face of the wedge and an impedance boundary condition on the other. The problem is approached by utilizing a transformation which reduces both boundary conditions to homogeneous ones and leaves the wave equation invariant. The solution to that problem is elementary; the transformation is then inverted and the solution examined in the limits of short and long wavelengths relative to the impedance parameter.

The amplitude of the surface wave propagated along the impedance boundary is compared to that when both surfaces have the impedance boundary condition. The ratio of the amplitudes in these two cases ranges from unity to  $\sqrt{2}$  between the two extreme wavelength limits mentioned previously. Finally a simple approximate expression for the far field excluding the surface wave is derived. *E. T. Kornhauser* (Providence, R.I.)

6396:

Getmancev, G. G. Growth of electromagnetic waves in interpenetrating infinite moving media. *Z. Eksper. Teoret. Fiz.* **37** (1959), 843-846 (Russian); translated as *Soviet Physics. JETP* **10** (1960), 600-602.

Author's summary: "An investigation is made of the propagation of monochromatic plane waves in interpenetrating moving media. Equations are obtained for the refractive index; these equations are used to investigate the stability of the propagating waves. The time growth (damping) factor for the wave is found for the case of motion of a plasma through a dispersionless dielectric."

*R. D. Kodis* (Providence, R.I.)

6397:

Kogelnik, Herwig. On electromagnetic radiation in magneto-ionic media. *J. Res. Nat. Bur. Standards Sect. D* **64D** (1960), 515-523.

Radiation in anisotropic media is considered and a wave matrix is defined, the zeros of its determinant giving the propagation constants of plane waves with a given normal. A dyadic Green's function is derived for the unbounded medium, such that the electric field is expressed as an integral over the current distribution. The relation between this dyadic function and the Green's tensor derived by F. V. Bunkin [*Z. Eksper. Teoret. Fiz.* **32** (1957), 377-379; *MR* **19**, 804] is noted. The mean complex power radiated is expressed as an integral involving the inverse of the wave matrix and the spatial Fourier transform of the current density and its Hermitian conjugate. The method is illustrated by a discussion of the real power radiated by an elementary dipole in a lossless medium of an ionized gas in a constant magnetic field.

*J. A. Morrison* (Murray Hill, N.J.)

6398:

Olving, Sven. A new method for space charge wave interaction studies. I. *Chalmers Tekn. Högsk. Handl.* No. 178 (1956), 12 pp.

A recognized procedure in studying the fundamental mechanism of the space charge wave interaction between a cylindrical electron beam and a concentric wire helix is to increase both the beam and helix radii indefinitely, keeping the difference finite. The helix structure then becomes a system of parallel, closely spaced straight wires and this is replaced by a plane sheet which has a finite conductivity in the direction of the wires. This paper simplifies the geometry even further by asserting that the fundamental properties of the system are not altered if an infinite number of such wire walls are placed parallel and close to each other, with the electrons passing between them. The mathematical model then becomes that of a three-dimensional medium which possesses a finite conductivity in a certain direction. The only geometry remaining is the angle  $\xi$  between the dc beam current (the

electrons being confined by an infinite dc magnetic field) and the direction of conductivity of the medium. (For  $\xi=0$  the medium is one used in connection with the resistive-wall amplifier.)

For the small pitch helix ( $\cos^2 \xi \approx 1$ ) and low dc current densities, the propagation constants of the TM waves have the same form as those for the helical type travelling wave tube (TWT). Quantitatively, there is a stronger interaction in the idealized case, as is to be expected. Other well-known TWT properties are qualitatively obtained. A correct picture with respect to velocity bandwidth is obtained only for the thick beam TWT (in which the electron beam completely fills the helix). The model becomes qualitatively different for sufficiently high dc current densities. J. A. Morrison (Murray Hill, N.J.)

6399:

Olving, Sven. A new method for space charge wave interaction studies. II. Chalmers Tekn. Högsk. Handl. No. 228 (1960), 40 pp.

The author continues his studies of space charge wave interaction between an electron beam (confined by an infinite dc magnetic field) and a medium which has a finite conductivity in a given direction. [See preceding review.] The system may be regarded as a plane travelling wave tube (TWT). The electron beam is now assumed to be of finite length, and reflection coefficients at the input and output ends are determined, under suitably approximated matching conditions. The input reflection coefficient is found to be surprisingly small under a variety of conditions, which would seem to justify the general assumption that this is so in practical cases. It is shown that at high current densities the backward electromagnetic wave and the fast space charge wave will form a pair of evanescent waves in the plane TWT, but that these evanescent waves are not present in the corresponding sheath helix TWT.

The author next considers the axially inhomogeneous plane TWT in which the dc electron velocity and the conductivity of the medium (in both direction and magnitude) vary along the beam. This problem reduces to the solution of a fourth order linear equation with variable coefficients. Neglecting coupling between the individual waves, this equation may be solved by WKB-type methods for slow variations along the beam. Furthermore, a set of four first order coupled equations is derived from which the effects of coupling between the waves can be estimated. An extension of this method leads to an important analysis of the transformation of non-growing waves into growing waves, as the direction of conductivity varies slowly, with distance, through a critical value.

A final section analyses the eight waves which can exist in a single-velocity magneto-ionic anisotropic homogeneous resistive medium. The so-called transverse space charge waves, the nature of which has been misinterpreted in previous work, are treated in detail.

J. A. Morrison (Murray Hill, N.J.)

6400:

Kurliko, V. I. Reflection of electromagnetic waves from moving surfaces. *Z. Tehn. Fiz.* 30 (1960), 504-507 (Russian); translated as Soviet Physics. Tech. Phys. 5, 473-476.

This paper describes a procedure for determining the

electromagnetic field in a non-dispersive medium contained between two parallel perfectly conducting planes which are approaching each other with equal and opposite constant velocities, the initial field being prescribed. When the velocity of the planes is less than the phase velocity of the medium, multiple reflections occur, and there is an increase with time in both the frequency and the amplitude of the fields between the planes.

J. A. Morrison (Murray Hill, N.J.)

6401:

Robieux, J. Lois générales de la liaison entre radiateurs d'ondes. Application aux ondes de surface et à la propagation. III. Propriétés des liaisons par diffraction et diffusion. *Ann. Radioélec.* 15 (1960), 331-377. (English and German summaries)

6402:

Garibyan, G. M. Transition radiation for a charged particle at oblique incidence. *Z. Eksper. Teoret. Fiz.* 38 (1960), 1814-1816 (Russian. English summary); translated as Soviet Physics. JETP 11, 1306-1307.

Author's summary: "The transition radiation emitted in the forward direction by a charge in oblique incidence at the boundary between two media is considered. It is shown that the intensity of the radiation is essentially independent of the angle of incidence of the particles so long as this angle is far from  $90^\circ$ ."

6403:

King, Ronald W. P.; Wu, Tai Tsun. ★The scattering and diffraction of waves. Harvard Monographs in Applied Science, No. 7. Harvard University Press, Cambridge, Mass., 1959. 218 pp. \$6.00.

The monograph is a good summary of research on diffraction and scattering carried out at Harvard University. While attention is directed mainly to electromagnetic aspects of the phenomena, many references are made to parallel problems in acoustics. As admitted by the authors, it is not a comprehensive study of the field, but rather a cross-sectional view of a few aspects of the subject.

The very difficult task confronted by the authors is to present a coordinated presentation of theory and experiment. Their attempt, in the opinion of this reviewer, is successful by and large. The only general criticism is that the subject is not placed in its proper historical perspective. For example, the early work of L. V. Lorenz, Lord Rayleigh, P. Debye, G. N. Watson, and F. P. White are not even mentioned.

The contents of the monograph include a short summary of the theory of the classical eigenfunction expansion for scattering by a circular cylinder of perfect conductivity. Extensive numerical results are presented for a cylinder whose diameter is comparable with the wavelength. A rather descriptive account of the theory of diffraction by obstacles of more general shape is also given. (For details the reader must obtain Cruft Lab. Reports No. 9 (1957) and No. 22 (1958), Harvard University.) The method which was developed at Harvard by Wu is based on the introduction of a new set of geometrical variables for the scatterer. Rather than the simple cylindrical coordinates  $(\rho, \theta_1)$  for the circular cylinder, the new variables  $u$  and  $t$

are  $u = \theta_1 - \frac{1}{2}\pi - \cos^{-1}(a/\rho)$  and  $t = (\rho^2 - a^2)^{1/2}/a$ . These are useful in the shadow region of the convex non-circular cylinder since  $a(u+t)$  is simply the optical distance from the point of observation  $(\rho, \theta_1)$  to the reference plane  $\theta_1 = \frac{1}{2}\pi$ . By making use of the boundary-layer theory of fluid dynamics and expressing the wave equation and the boundary conditions in terms of the new variables, the solution for the shadow region is obtained. The result is in the form of an asymptotic series of which the leading term depends only on the local radius of curvature of the convex object. This first term can also be predicted by J. B. Keller's geometrical theory of diffraction.

The current induced on an elliptic cylinder is also discussed in some detail. The method, which is attributed to Wetzel, divides up the surface into small segments over each of which the radius of curvature may be considered approximately constant. The local behavior of each segment is calculated as if it were a section of a circular cylinder. The method appears to be quite similar to those developed by Keller (for the umbra) and Fock (for the penumbra). Higher order terms in Wetzel's formula for the surface current involve the rate of change of curvature. It is a pity that no attempt is made to compare Wetzel's and Wu's theories applied to elliptic cylinders. They should be mutually valid in the shadow region.

Other topics discussed in the monograph include scattering from disks, strips, and loops, transmission through slits and apertures, and scattering of waves that are not plane. Experimental techniques for measuring back-scattering cross section and diffracted fields of obstacles are also described in a lucid fashion.

J. R. Wait (Boulder, Colo.)

6404:

Neugebauer, H. E. J.; Bachynski, M. P. Diffraction by smooth conical obstacles. J. Res. Nat. Bur. Standards Sect. D 64D (1960), 317-329; errata, 589.

Kirchhoff's theory with suitable modifications is used to calculate diffraction by smooth conical mountains in the case of oblique incidence. The method is an extension of that used previously by the same authors in treating diffraction by smooth cylinders. The integrals obtained are evaluated by the stationary phase principle, and the results obtained appear to agree well with experiments performed with scale models.

E. W. Marchand (Rochester, N.Y.)

6405:

Gurevič, L. È.; Pavlov, S. T. Scattering of electromagnetic waves on free electrons in a strong magnetic field. Ž. Tehn. Fiz. 30 (1960), 41-43 (Russian); translated as Soviet Physics. Tech. Phys. 5, 37-39.

This is a short paper which presents the results of a quantum mechanical perturbation calculation of the scattering cross section for electromagnetic waves incident on free electrons in a strong magnetic field. Explicit formulas are given for the two limiting cases,  $ka \ll 1$  and  $ka \gg 1$ , where  $a = (c\hbar/eH_0)^{1/2}$ .

R. D. Kodis (Providence, R.I.)

6406:

Phariseau, P. The diffraction of light by an amplitude modulated ultrasonic beam. Physica 25 (1959), 917-923.

In this paper the author treats the problem of propagation of an oblique incident monochromatic light wave

in a homogeneous medium perturbed by an amplitude modulated ultra-acoustic wave of frequency  $\nu^0$ . This problem leads to the solution of the Helmholtz equation in two variables in a medium (slab of finite thickness and infinite extent along the direction of propagation of the acoustic wave,  $x$ ) with a periodic refractive index in  $t$  and  $x$ . Assuming the variation of the refractive index inside the slab to be small in comparison to that of the homogeneous medium and small deviations in the curvature of the light beam within the slab, a system of first order non-homogeneous equations is derived. Without going into the details of the calculations, the effect of first order perturbations gives rise to a splitting of various order diffracted beams. The effect of the modulation is to split the first order diffracted spectra into three components, symmetric about the modulating frequency  $\nu^*$ , ( $\nu^0 \pm \nu^*$ ,  $\nu^*$ ). On the other hand the diffracted lines which correspond to the modulating frequency do not appear in this case, i.e., the zero order spectrum is not affected by the perturbation. The intensities of the lines of order  $(1, 0; 0, 1)$  are proportional to the square of the non-modulated amplitude of the refractive index, whereas those of the order  $(1, 1; 1, -1; -1, 1; -1, -1)$  are proportional to the square of the modulated amplitude. Finally, the location of the maximum intensity of each diffracted line is given by Bragg's Equation corresponding to these lines.

N. Chako (Flushing, N.Y.)

6407:

Phariseau, P. Diffraction of light by a three-dimensional system of ultrasonics. Physica 25 (1959), 924-934.

The author extends the results of the previous article (see above) to the case where the acoustic beams, mutually independent, and orthogonal to each other are propagated within a slab of thickness  $d$ . As in the former problem, the refractive index within the slab is made up of a large constant term plus three periodic terms in  $t$  and  $r$  (vector coordinate) with small constant amplitudes. The author bases his analysis on the ordinary Lorentz equation for the electric field  $E$ . By expanding  $E$  in the form

$$(1) \quad E = \sum_K \varphi(r, t, |K|) \exp(i\omega t)$$

( $K$  is the wave vector,  $|K| = \mu_0 \omega / c$ ,  $\omega$  = frequency of the light incident waves) and making plausible approximations, each  $\varphi_K$  satisfies an equation

$$(2) \quad \Delta \varphi + \omega^2 (\mu/c)^2 \varphi = 0,$$

where  $\mu$  is the refractive index inside the slab. Assuming  $\varphi$  to be of the form

$$(3) \quad \varphi(r, t, |K|) = \sum_{lms} N_{lms}(|K|) \exp i(k_{lms} - \gamma + \omega_{lms}^* + \delta_{lms}),$$

$k_{lms} = k - k_{lms}^*$ ,  $k_{lms}^* = lk_1^* + mk_2^* + nk_3^*$  ( $k_j$  are the wave vector components of the acoustic waves), a system of homogeneous equations for the determination of  $k$  and the amplitudes  $N_{lms}$  is obtained. Letting  $k = K + \sigma u$ , the system leads to a determinant relation for calculating  $\sigma$  and the amplitudes  $N_{lms}$ . From the form of the field  $E$  within and outside the slab and the boundary conditions, a set of equations is derived for determining the refracted

and transmitted fields. Expanding the amplitudes  $N_{lms}$  and  $\sigma$  in ascending powers of the parameter

$$\eta = \frac{1}{2}\mu_0^{-1}(\mu_1 + \mu_2 + \mu_3)$$

( $\mu_0, \mu_1, \mu_2, \mu_3$  are the constant term and the amplitudes of the perturbed refractive index), the zero and the first order approximation of the refractive and transmitted fields are calculated. The results of the analysis are shown in a table, where the splitting of the diffracted lines of zero and higher order spectra are indicated. [The reviewer wishes to point out that it is not quite correct to neglect the terms in the derivatives of the field  $E$ , when deriving the wave equation from Maxwell equations, since these terms would contribute effects which would be of the same order as those given by first order perturbation terms, unless it is shown that such terms do not contribute to the first order approximation. This the author has not done in the text. There are also several misprints, among which the most serious are the following. In equations (10) and (12) of the text,  $\mu_0$  should be replaced by  $\mu$ , and in (19),  $|K|$  should be replaced by  $|K|^2$  and  $\mu$  by  $u$ .]

N. Chako (Flushing, N.Y.)

6408:

King, Ronald; Harrison, Charles W. Half-wave cylindrical antenna in a dissipative medium: current and impedance. J. Res. Nat. Bur. Standards Sect. D **64D** (1960), 365-380.

Authors' summary: "An integral equation for the distribution of current along a cylindrical antenna in a conducting dielectric is derived. It is shown that the boundary conditions for an antenna in such a medium are formally the same as for an antenna in free space. The equation is solved for the current  $I$  and the driving-point impedance  $Z$  by means of a technique that achieves sufficiently high accuracy in the leading terms of an iteration procedure so that the higher-order terms do not need to be evaluated. Moreover, these leading terms consist only of trigonometric functions with complex coefficients. The electromagnetic field in the infinite dissipative medium may be computed relatively easily since the current in the antenna is expressed in such simple terms.

"A numerical analysis is made to determine the properties of an antenna with an electric length of one-half wavelength in the medium with conductivity  $\sigma$  and relative dielectric constant  $\epsilon_r$ . Universal curves are given of  $I/\sqrt{\epsilon_r}$  with  $\sigma/\omega\epsilon_0\epsilon_r$  as the parameter and of  $Z/\sqrt{\epsilon_r}$  with  $\sigma/\omega\epsilon_0\epsilon_r$  as the variable in the range  $0 \leq \sigma/\omega\epsilon_0\epsilon_r \leq 0.4$ . A table of numerical values of the impedance is given for media such as an isotropic ionosphere, dry salt, dry earth, wet earth, and lake water."

G. Sinclair (Toronto)

6409:

Duncan, R. H.; Hinchey, F. A. Cylindrical antenna theory. J. Res. Nat. Bur. Standards Sect. D **64D** (1960), 569-584.

Authors' summary: "A partial survey of cylindrical antenna theory pertaining to a tubular model with a narrow gap is presented. The survey includes discussion of the theories of Hallén, King and Middleton, Storm, and Zuhrt. A conceptual relation between theory and experiment is described. The latter part of the article is concerned with a new Fourier series solution of the Hallén equation. This solution is developed in such a way that

the expansion coefficients are the unknowns of a system of linear equations. The elements of the coefficient matrix are given by a highly convergent series. Numerical results are given for half and full wavelength antennas with half length to radius ratios of 60 and 500. These results compare quite closely with those obtained from King-Middleton theory."

G. Sinclair (Toronto)

6410:

Toraldo di Francia, Giuliano. Trasporto di momento angolare in una guida d'onda. Alta Frequenza **29** (1960), 148-153.

Author's summary: "Il momento angolare trasportato dal campo elettromagnetico all'interno di una guida d'onda viene calcolato per mezzo delle azioni meccaniche esercitate dall'onda. Nel caso che il campo risulti dalla sovrapposizione di due o più modi degeneri, si può ammettere che il dielettrico presenti una piccolissima conduttività e calcolare il momento esercitato dal campo sulle correnti che nascono nel mezzo. Si trova che il momento angolare viaggia con la stessa velocità dell'energia. Nel caso di due modi non degeneri, si ha un continuo scambio di momento angolare fra il campo e le pareti della guida. Risulta che il momento angolare si sposta con una velocità che è la media aritmetica delle velocità di gruppo dei due modi."

6411:

Nicolau, Edmond. On the theory of receiving antennas systems. Bul. Inst. Politehn. București **20** (1958), no. 4, 165-170. (Russian, French and German summaries)

6412:

Brehovskikh, L. Propagation of acoustic and infrared waves in natural waveguides over long distances. Uspehi Fiz. Nauk **70** (1960), 351-360 (Russian); translated as Soviet Physics. Uspekhi **3**, 159-166.

6413:

Barlow, H. E. M. Non-reflecting waveguide tapers. Proc. Inst. Elec. Engrs. B **107** (1960), 515-521.

6414:

Nicolau, Edmond. On the network equations in antenna theory. Bul. Inst. Politehn. București **20** (1958), no. 3, 153-159. (Russian, French and German summaries)

A formula for the mutual impedance between two arbitrary antennas is obtained. G. Sinclair (Toronto)

6415:

Merkulov, V. V. Field structure in cylindrical waveguides with a complex cross section. Akust. Zh. **5** (1959), 428-431 (Russian); translated as Soviet Physics. Acoust. **5** (1960), 439-443.

Fields in uniform waveguides having cross-sectional shapes represented by perturbation from a simple shape are considered. The wave equation is transformed by a conformal transformation to a wave equation with variable coefficients in a region having simple boundaries. The transformation used is of the form

$$x + iy = \zeta + h\Phi(\mu\zeta),$$

where  $\zeta = \xi + i\eta$  and  $\Phi$  is analytic. The transformed wave equation depends on the parameter  $\varepsilon = h\mu$ , so it can be solved by perturbation theory when  $|\varepsilon| \ll 1$ . It can be seen that  $|\varepsilon| \ll 1$  not only when  $h \ll 1$  (i.e., small deformation of the cross-sectional shape) but also when  $h \gg 1$  (large deformation) if  $\mu \ll 1$ . The author treats only the example  $\Phi(\mu\zeta) = \cos \zeta$ , and does not show a case where  $\mu \ll 1$  produces a large deformation of the cross-sectional shape.

G. Sinclair (Toronto, Ont.)

6416:

Simkin, M. M. Periodic modes in systems with non-linear pulse elements: Dokl. Akad. Nauk SSSR 131 (1960), 1323-1326 (Russian); translated as Soviet Physics. Dokl. 5, 227-230.

A loop consisting of an element generating a periodic pulse train, a non-linear amplitude, width, and/or position modulating element, and a linear passive two-port with low-pass characteristic, but no d.c. passage, is considered. The generating element opens the loop during pulse time; at other times the loop is closed. Self-oscillating modes if extant are subharmonic of the pulse frequency. An expansion method, based on harmonic balance, with a describing function approach in lowest order of approximation, is presented for mode determination in the most general case. As both expansion of the modulation characteristic and frequency domain treatment are involved, the method necessarily produces implicit relationships, and recourse to graphical procedures is indicated at some point in application.

H. G. Baerwald (Albuquerque, N.M.)

6417:

Millman, Jacob; Taub, Herbert. ★Pulse and digital circuits. McGraw-Hill Electrical and Electronic Engineering Series. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1956. xvi + 687 pp. \$12.50.

This textbook covers aspects of electronics usually dealt with in senior and first year graduate courses in electrical engineering. In the first four chapters (140 pp.) the linear and piecewise linear models of electronic circuits are described and applied to the discussion of a number of standard circuits. The remainder of the book is organized into chapters on the numerous circuits that are used in sweep generation, pulse formation, pulse transmission, counting, logic, etc. The mode of operation of these circuits and their application is discussed in considerable engineering detail.

R. D. Kodis (Providence, R.I.)

6418:

Guillemin, E. A. The normal coordinate transformation of a linear system with an arbitrary loss function. J. Math. and Phys. 39 (1960/61), 97-104.

Here is a novel treatment of topological lumped electrical networks, employing compound matrix representation to effect reduction to normal coordinate form. With the parameter matrices  $L = (l_{ks})$ ,  $C = (c_s)$ ,  $R = (r_s)$ ,  $G = (g_s)$ , the product  $\gamma_{ln}$  of the associated tie-set and transposed out-set matrices, and the column vectors  $i$ ,  $e$ ,  $e_s$  and  $i_s$  of loop current, mode-pair voltage variables, voltage and current sources, respectively, the equilibrium equations can be written:

$$\left\{ \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix} p + \begin{bmatrix} R & \gamma_{ln} \\ -\gamma_{nl} & G \end{bmatrix} \right\} \times \begin{bmatrix} i \\ e \end{bmatrix} = \begin{bmatrix} e_s \\ i_s \end{bmatrix}.$$

If the parameter matrices are positive definite, the symmetrical compound matrix pertaining to

$$\left[ \frac{(\sqrt{-1})i}{e} \right] \text{ and } \left[ \frac{(\sqrt{-1})e_s}{i_s} \right]$$

can be congruently reduced to diagonal form. Energy and power are conserved, and the diagonal consists of the complex natural frequencies. For coherent exponential excitation, an associated immittance matrix ensues, with the residues of the elemental partial fractions simply related to the transformation matrix. In the more general case where  $G$  and  $R$  may be active, diagonalization without power invariance is achieved by tandem application of a real congruent and a complex co-linear transformation. This representation is pertinent to direct synthesis procedures encompassing equivalences.

H. G. Baerwald (Albuquerque, N.M.)

#### CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 6085, 6086, 6217.

6419:

Weinstein, M. A. Thermodynamics of radiative emission processes. Phys. Rev. (2) 119 (1960), 499-501.

Author's summary: "A basic assumption implicit in the application of thermodynamics to the electromagnetic field is that the laws of thermodynamics are locally valid for radiative emission and absorption processes. This means that a certain minimum amount of entropy must be created by the radiative process itself. It is shown, by considering the extreme case in which the spontaneous emission of a natural spectral line is the only process taking place, that this assumption is correct, and that its validity is essentially a consequence of the uncertainty principle as expressed by the reciprocal relationship between natural line breadth and lifetime."

6420:

Chu, W. H.; Abramson, H. N. Transient heat conduction in a rod of finite length with variable thermal properties. J. Appl. Mech. 27 (1960), 617-622.

Authors' summary: "This paper presents a theoretical solution for transient heat conduction in a rod of finite length with variable thermal properties. A numerical procedure is developed and the results of one example are presented and compared with the corresponding solution for the case of constant properties. Application to the problem of determination of thermophysical properties is discussed briefly."

6421:

Donnadieu, Gérard A. Étude des échanges thermiques entre fluide et particules solides dans les milieux fluidisés. J. Rech. Centre Nat. Rech. Sci. No. 51 (1960), 161-169.

6422:

Fujita, Shigeichi. On the stationary flow of vapor. Kumamoto J. Sci. Ser. A 4, 210-222 (1960).

6423:

Nettleton, R. E. Relaxation theory of thermal conduction in liquids. *Phys. Fluids* 3 (1960), 216-225.

Author's summary: "A linear relaxation equation for the heat flux in a fluid, proposed by Vernotte as a generalization of Fourier's law, is shown for liquids to be consistent with the assumption that thermal energy is carried by elastic waves of very high frequency which may be envisaged as being propagated in a continuum. The elastic constants and the velocity of the waves are obtained from the infinite frequency limits of viscoelastic equations, derived in earlier papers to describe the relaxation of compressional and shearing strains, and from these, the relaxation time and thermal conductivity are calculated for several nonassociated liquids with the aid of a theory of Debye. It is shown that the Vernotte equation may be viewed formally, from the point of view of irreversible thermodynamics, as a force-flux equation linking two irreversible processes, and this interpretation makes it possible to calculate terms in the pressure and internal energy which are nonlinear in the temperature gradient." H. L. Frisch (Murray Hill, N.J.)

6424:

Ribaud, G. Conduction de la chaleur en régime variable. *Mémor. Sci. Phys.* 65 (1960), 90 pp.

This booklet treats a number of simple problems in conduction of heat by mathematical procedures that are kept on or near the level of elementary calculus. Solutions of one-dimensional flow problems are written by assuming particular forms of solutions of the heat equation and verifying them. Media consist of slabs, bars and, in a few cases, cylinders or spheres. The nature and physical significance of the results are discussed with the aid of graphs. Finite difference and graphical methods are illustrated briefly. R. V. Churchill (Ann Arbor, Mich.)

6425:

Vodička, Václav. Steady temperature field in a composite doubly infinite strip. *J. Phys. Soc. Japan* 15 (1960), 1332-1336.

Let  $u_k(x, y)$  denote steady temperatures in the layer  $x_{k-1} \leq x \leq x_k$  ( $k=1, 2, \dots, n$ ) of a composite slab bounded by two planes  $x=x_0$  and  $x=x_n$ . If  $\lambda_k$  is the thermal conductivity of the material in the  $k$ th layer and if the layers are in perfect thermal contact, then  $u_k(x, y)$  satisfies Laplace's partial differential equation interior to the  $k$ th layer, the boundary conditions  $u_1(x_0, y)=f(y)$ ,  $u_n(x_n, y)=g(y)$ , and interface conditions  $u_{k+1}=u_k$  and  $\lambda_k \partial u_k / \partial x = \lambda_{k+1} \partial u_{k+1} / \partial x$  at  $x=x_k$  ( $k=1, 2, \dots, n-1$ ), where  $f$  and  $g$  are prescribed functions. Using exponential Fourier transforms with respect to  $y$  ( $-\infty < y < \infty$ ) the author outlines the formal process of solving for  $u_k(x, y)$ . Further details are given in the case  $n=2$ .

R. V. Churchill (Ann Arbor, Mich.)

6426:

Short, W. W. Heat transfer and sublimation at a stagnation point in potential flow. *J. Appl. Mech.* 27 (1960), 613-616.

Author's summary: "A simple analytical expression is derived which predicts the effect of mass transfer on countercurrent heat transfer to a vaporizing body. In this theory, the fluid stream is assumed to be inviscid and of

constant thermal conductivity. The inviscid theory correlates well with heat-transfer data without mass-transfer and is believed to predict heat-transfer rates fairly accurately at high mass-transfer rates."

6427a:

Vodička, Václav. Fourier's classical problem in the case of stratiform bodies. *Arch. Mech. Stos.* 12 (1960), 3-12. (Polish and Russian summaries)

6427b:

Vodička, Václav. Steady temperature in a composite semi-infinite three-layer plate. *Arch. Mech. Stos.* 12 (1960), 151-162. (Polish and Russian summaries)

Let  $u_k(x, y)$  denote steady-state temperatures in each of  $n$  layers  $x_{k-1} \leq x \leq x_k$ ,  $y \geq y_0$  ( $k=1, 2, \dots, n$ ) of the semi-infinite solid  $x_0 \leq x \leq x_n$ ,  $y \geq y_0$ ,  $-\infty < z < \infty$ . Let  $\lambda_k$  denote the constant thermal conductivity of the  $k$ th layer, and  $f_k(x)$  the temperature of the boundary  $y=y_0$  of that layer. If the outer faces are kept at temperature zero, the functions  $u_k(x, y)$  satisfy this boundary value problem:  $\nabla^2 u_k = 0$  interior to the  $k$ th layer,  $u_1(x_0, y)=0$ ,  $u_n(x_n, y)=0$ ,  $u_k(x, y_0)=f_k(x)$ , and interface conditions for perfect contact, namely,  $u_k=u_{k+1}$  and  $\lambda_k \partial u_k / \partial x = \lambda_{k+1} \partial u_{k+1} / \partial x$  at  $x=x_k$  ( $k=1, 2, \dots, n-1$ ). Separation of variables gives a discontinuous eigenvalue problem on the interval  $(x_0, x_n)$ . Orthogonality relations for the eigenfunctions, and equations that determine the eigenvalues and eigenfunctions are indicated, so that a formal solution for  $u_k(x, y)$ , bounded for large  $y$ , can be written in series form. Further details are carried out in special cases when  $n=2$  or  $n=3$ .

R. V. Churchill (Ann Arbor, Mich.)

6428:

Lewis, J. A.; Riney, T. D. Temperature rise in an infinite medium heated by a planar source. *J. Soc. Indust. Appl. Math.* 8 (1960), 249-271.

Let  $u(P, t)$  denote the temperatures at time  $t$  at points  $P: (x, y, z)$  of a medium of infinite extent in all directions. The medium, initially at temperature zero throughout, is heated by a source of intensity  $Q(t)$  per unit area distributed uniformly over a bounded domain  $D$  of area  $A$  in the  $xy$  plane, where  $Q(t) \geq 0$ . Beginning with the integral representation of  $u(P, t)$  in terms of Green's function, using the Laplace transform  $\bar{u}(x, s)$  and connections between the asymptotic behavior of  $u$  and  $\bar{u}$  for large and small  $t$  and small and large  $s$ , the authors obtain considerable information on upper bounds of  $u(P, t)$ .

Of all domains  $D$  with common area  $A$  and common source intensity  $Q(t)$ , the circular domain yields the greatest temperature  $u(P, t)$ . The plane  $z=0$  contains the points of maximum temperature. If  $D$  is convex, or if  $Q(t)$  is a nondecreasing function, then the maximum of  $u$  occurs within  $D$ . The Steiner symmetrization of  $D$  is used to show that, if  $D$  has a line  $L$  of symmetry such that lines perpendicular to  $L$  cut the boundaries of  $D$  in at most two points, then  $L$  contains a point where  $u$  is maximum. Upper bounds of  $u$  are found by replacing  $D$  by a circular domain. Asymptotic expressions for  $u$  provide simple estimates of  $u$  when  $t$  is either large or small. Estimates are also found for steady-state temperatures, when  $\lim_{t \rightarrow \infty} Q(t)$  exists. Lower bounds on  $u(P, t)$  are considered briefly.

R. V. Churchill (Ann Arbor, Mich.)

6429:

Hirschfelder, Joseph O.; Curtiss, Charles F. The theory of flame propagation. Appl. Mech. Rev. 13 (1960), 231-234.

6430:

Carnot, Sadi. ★Reflections on the motive power of fire. And other papers on the second law of thermodynamics by É. Clapeyron and R. Clausius. Edited with an introduction by E. Mendoza. Dover Publications, Inc., New York, 1960. xxii + 152 pp. \$1.50.

"Reflections on the motive power of fire, and on machines fitted to develop that power," by Sadi Carnot is an unabridged and slightly corrected republication of *Reflections on the motive power of heat* by Sadi Carnot, translated and edited by R. H. Thurston, and published by Macmillan and Company in 1890. A new appendix, "Selections from the Posthumous Manuscripts of Carnot", translated by R. H. Thurston and E. Mendoza, is added.

"Mémorial on the Motive Power of Heat" by É. Clapeyron was especially translated for this edition by E. Mendoza.

"On the motive power of heat, and on the laws which can be deduced from it for the theory of heat" by R. Clausius was translated by W. F. Magie, and originally appeared in the volume entitled *The Second Law of Thermodynamics*, edited by W. F. Magie, and published by Harper and Brothers in 1899.

6431:

Garsia, A. Some models of explosive line detonators. J. Math. and Phys. 39 (1960/61), 54-57.

This note is concerned with the shape of a sheet of explosive such that if detonation is started at some point  $F$  on the sheet, at some later time the detonation front lies along a straight line  $AB$ . This resolves to the geometrical problem of the geodesic path along the surface from any point on  $AB$  to  $F$  being of constant length. There are many ways of achieving this and the author discusses models which are convenient for practical applications.

H. Kolsky (Providence, R.I.)

6432:

Keck, James C. Variational theory of chemical reaction rates applied to three-body recombinations. J. Chem. Phys. 32 (1960), 1035-1050.

Author's summary: "A 'variational' theory, which gives a least upper bound to the rate of a chemical reaction, is presented. The reaction is represented by the motion of a point in phase space across a trial surface dividing the 'initial' and 'final' chemical states. The trial surface is well defined in regions of phase space where interactions causing reaction are negligible, but is subject to arbitrary variations otherwise. It is shown that a least upper bound to the reaction rate can be obtained by calculating the rate at which representative points cross the 'trial' surface and then minimizing this rate with respect to allowed variations of the surface. Explicit calculations of the recombination rate of attracting atoms in the presence of repulsive third bodies are made for a simple trial surface having one adjustable parameter. At low temperatures, the experimental rate constants are quite close to the theoretical bounds; at high temperatures, the experi-

mental data fall away from the bounds in a manner which can be understood in terms of various approximations contained in the theory. Promising methods of improving the agreement between theory and experiment are discussed."

H. L. Frisch (Murray Hill, N.J.)

6433:

Rice, O. K. Conditions for a steady state in chemical kinetics. J. Phys. Chem. 64 (1960), 1851-1857.

Author's summary: "A general set of inequalities involving kinetic quantities is derived, whose validity is necessary for the establishment of a steady state in a reaction involving transient species. The stability of the steady state requires further that all the roots of a certain determinantal equation be negative (or, if complex, have a negative real portion). These criteria are applied to a number of examples, including branching chain reactions, the hydrogen bromide reaction, an oscillating reaction, a set of consecutive reactions, and a mechanism of interest in the consideration of flow processes, and the various factors affecting their application are elucidated."

## QUANTUM MECHANICS

See also 6134, 6135, 6136, 6137, 6585.

6434:

Fabri, E. The logical foundations of invariance principles in physics. I. Nuovo Cimento (10) 14 (1959), 1130-1144. (Italian summary)

The paper discusses in detail some of the notions which appear in the formulation of invariance principles: reference frame, coordinate system, relation between the descriptions of the same fact in two coordinate systems, etc. Two reference frames  $R_1$  and  $R_2$  are said to be coherent if (a) a fact is describable in  $R_1$  if and only if it is describable in  $R_2$ , and (b) a description has a sense in  $R_1$  if and only if it has a sense in  $R_2$ . A description in a reference frame  $R$  is said to be reliable in  $R$  if the fact which it describes is true. Two reference frames are equivalent if every description reliable in one is reliable in the other. The general principle of equivalence can then be stated: all coherent reference frames are equivalent. From this the author deduces: any reliability criterion for a description must be invariant under the group expressing the equivalence of the descriptions in different reference frames. The paper concludes with a comparison of his work with an unpublished manuscript of Wigner, Bargmann and the reviewer.

A. S. Wightman (Princeton, N.J.)

6435:

Freund, P. G. O.; Hegedüs, I. Zur gruppentheoretischen Begründung der Quantenmechanik. I, II. Acta Phys. Acad. Sci. Hungar. 11, 285-289, 291-294 (1960).

It is indicated mainly that: (I) the Schrodinger equation with a spherically symmetric potential may be characterized as the general linear partial differential equation of 2nd order in the space variables and first order in time that transforms suitably under rotations in space; (II) the generators or rotations are essentially the angular momenta of a field.

I. E. Segal (Cambridge, Mass.)

6436:

Širokov, Yu. M. A group-theoretical consideration of the basis of relativistic quantum mechanics. V. Ž. Eksper. Teoret. Fiz. **36** (1959), 879-888 (Russian); translated as Soviet Physics. JETP **9**, 620-626.

[For part IV, see same Ž. **34** (1958), 717-724; MR **21** #2483.]

The author constructs eight covering groups of the inhomogeneous Lorentz group including the inversions,  $I_1(x^0 \rightarrow x^0, \mathbf{x} \rightarrow -\mathbf{x})$ ,  $I_2(x^0 \rightarrow -x^0, \mathbf{x} \rightarrow \mathbf{x})$  and  $I_{34}(x \rightarrow -x)$ . The eight groups are distinguished by a choice of the identity or  $I_{23}$  for the square of  $I_1$ ,  $I_2$ , and  $I_{34}$ . Here  $I_{23}$  is the "rotation through  $2\pi$ " which lies above the identity element in the covering group of the connected part of the inhomogeneous Lorentz group. He then finds all unitary irreducible representations of these groups. (In the first investigation of this kind [E. P. Wigner, Ann. of Math. (2) **40** (1939), 149-204], the inversions were also taken unitary, but representations up to a factor of the inhomogeneous Lorentz group were considered and not regarded as different if they could be obtained from one another by a phase change.) The full classification is too complicated to summarize here. The author notes the form invariance of equations in quantum theory under unitary time inversion operators is only possible because the equations have solutions with both signs of the mass (unquantized Dirac equation) or because of an infinite vacuum background (quantized scalar and Dirac equations). Compare the following review of the paper by the same author in which time inversion is taken anti-unitary and this negative energy difficulty does not arise.

A. S. Wightman (Princeton, N.J.)

6437:

Širokov, Yu. M. Space and time reflections in relativistic theory. Nuclear Phys. **15** (1960), 1-12.

As in the paper treated in the preceding review the author introduces eight covering groups of the inhomogeneous Lorentz group and looks for irreducible representations, this time with anti-unitary time inversion. He also assumes that for particles of integer spin the squares of all reflections are equal to unity. For particles of half odd integer spin they may be  $\pm 1$  but must be the same for all particles. (The physical reason for these a priori requirements escapes the reviewer.) The classification turns out to be very complicated, e.g., for spin  $1/2$  there are 26 distinct representations. The author then tries to find out which of the representations are compatible with (a) odd parity of the ground state of positonium; (b) two component neutrino; (c) four components for all known spinor particles with non-zero mass; (d) PCT conservation; (e) CP conservation; (f) T conservation. He is led to a nearly unique covering group which he then interprets as the symmetry group of space-time.

A. S. Wightman (Princeton, N.J.)

6438:

Wakita, Hitoshi. Measurement in quantum mechanics. Progr. Theoret. Phys. **23** (1960), 32-40.

The author presents his views on the theory of measurement. They are in substantial but not complete agreement with those of H. S. Green [Nuovo Cimento (10) **9** (1958), 880-889; MR **20** #6930]. "In the course of 'measurement' the micro-system interacts with a measuring apparatus, and the prepared meta-stable state of the apparatus

changes into a new state, which is a superposition of several stable states. As these states should differ macroscopically, we can treat the new state as a mere probability function, and so, when we 'observe' it, it changes into one of these stable states abruptly and non-causally." The author supports this view with a discussion of the notion of state for systems for large and also for an infinite number of degrees of freedom.

A. S. Wightman (Princeton, N.J.)

6439:

Kurdgelaidze, D. F. Theory of the nonlinear field ( $\square - \lambda \varphi^2$ ) $\varphi = 0$ . Ž. Eksper. Teoret. Fiz. **36** (1959), 842-849 (Russian); translated as Soviet Physics. JETP **9**, 594-598.

Author's summary: "The nonlinear field described by the equation ( $\square - \lambda \varphi^2$ ) $\varphi = 0$  is considered. Starting from the exact wave-type solution of the field equation, the spectral analysis of the energy of the nonlinear field is obtained. The mass spectrum derived has the form  $M_0^{(n)} = (2n+1)M_0^{(0)}$ ,  $n=0, 1, 2, \dots$ . The exact radially symmetric solution of the field equation is found. A general method of integrating the nonlinear field of Dirac is given, and it is shown that in some cases it is possible to go over to a two-component spinor equation of the second order."

6440:

Shimpuku, Taro. General theory and numerical tables of Clebsch-Gordan coefficients. Prog. Theoret. Phys. Suppl. No. 13 (1960), 1-135. (4 inserts)

A new general expression for the Clebsch-Gordan coefficients is derived using the spinor representations of the three-dimensional rotation group. (The C-G coefficients are the matrix elements of the unitary similarity transformation which reduces the direct product of two irreducible representations of the rotation group.) Denoting the dimensions of the latter by  $(2j+1)$  and  $(2j'+1)$  respectively, algebraic tables are given for arbitrary  $j$  and  $j' = 7/2$  through 5 (including half-integral values), and numerical tables for  $j=5, 11/2, 6$  and  $j'=1/2$  through 6. These extend tables already published for physical applications in combining angular momenta in quantum mechanics.

D. Falkoff (Waltham, Mass.)

6441:

Dupont-Bourdelet, Françoise; Tillieu, Jacques; Guy, Jean. Sur le calcul des énergies perturbées d'ordre quelconque. J. Phys. Radium **21** (1960), 776-778. (English summary)

Authors' summary: "The perturbation energy of  $n$ th order can be calculated when the perturbation functions are known up to the order  $n/2$  for  $n$  even, or  $(n-1)/2$  for  $n$  odd. This is a generalization of a former result obtained by Dalgarno and Stewart [Proc. Roy. Soc. London Ser. A **238** (1956), 269-275; MR **18**, 851].

6442:

Lepore, Joseph V. Commutation relations of quantum mechanics. Phys. Rev. (2) **119** (1960), 821-826.

The paper is an elementary exposition of a number of ideas centering around the galilean invariance of non-relativistic quantum mechanics and its relation to the

commutation relations of position and coordinate. In particular, the relation between momentum and infinitesimal space translation and between position and infinitesimal Galilei transformation is explained.

A. S. Wightman (Princeton, N.J.)

6443:

Epstein, Saul T. Application of the Rayleigh-Schrödinger perturbation theory to the delta function potential. *Amer. J. Phys.* **28** (1960), 495-496.

Author's summary: "E. P. Wigner has shown that although there are reasons why one might hope to succeed, one cannot in fact derive the binding energy of the hydrogen atom by second-order perturbation theory applied to a free particle in a box. To further clarify the reasons for this failure we here consider the one-dimensional attractive delta function potential."

6444:

Radok, J. R. M. Transcendental equation for the Schrödinger equation. *Math. Comput.* **14** (1960), 276-278. A table of the first few roots of

$$\mathfrak{h}_\lambda^{(1)}(i - \sqrt{K^2 - \varepsilon^2}) + n_\lambda(\varepsilon) = 0,$$

where  $\mathfrak{h}_\lambda^{(1)}$  and  $n_\lambda$  are modified quotient Bessel functions, for  $\lambda = 0(1)3$ ,  $K = 1(1)15$ . The roots were obtained graphically and are given to 3 figures. D. F. Mayers (Oxford)

6445:

Schwinger, Julian. The geometry of quantum states. *Proc. Nat. Acad. Sci. U.S.A.* **46** (1960), 257-265.

This is the second of a series of papers in which the author builds from the basic laws of microscopic measurement the general structure first of quantum statics and then of quantum dynamics. The first [same *Proc.* **45** (1959), 1542-1553; MR **22** #3449] describes the algebra of "measurement symbols" such as  $M(b, a)$ , which denotes a process whereby a system is accepted only from a state in which a property A has the value  $a$  and emerges in a state in which B has the value  $b$ . In the paper under review the author introduces symbols representing states by remarking that, since the process denoted by  $M(b, a)$  is "physically" inseparable into successive stages, one is free to perform the "formal" factorization  $M(b, a) = M(b, 0)M(0, a)$ , where 0 denotes a fictitious null state. Then  $\psi(b) = \det M(b, 0)$  and  $\phi(a) = \det M(0, a)$ , as a consequence of the laws of the measurement algebra and the null property of 0, belong to two mutually adjoint vector algebras associated with the states of the system; the geometry of states is a unitary geometry. The elements of the measurement algebra receive a geometrical interpretation as linear operators, which themselves form a vector space of dimensionality equal to the square of that of the state space.

P. W. Higgs (Edinburgh)

6446:

Schwinger, Julian. Unitary operator bases. *Proc. Nat. Acad. Sci. U.S.A.* **46** (1960), 570-579.

Continuing the programme described in the preceding review, the author discusses the generation of a complete orthonormal operator basis from a "complementary" pair

of unitary operators. If in a state geometry of dimensionality  $N$  one defines  $U, V$  to be cyclic permutation operators such that  $\langle a^k | V = \langle a^{k+1} |$ ,  $\langle b^k | U^{-1} = \langle b^{k+1} |$ , modulo  $N$ , where  $\langle a^k |$ ,  $\langle b^k |$  ( $k = 1, \dots, N$ ) are two complete orthonormal bases in state space, the eigenvectors of  $U, V$  respectively, then the set  $X(m, n) = \det N^{-1/2} U^m V^n$  ( $m, n = 0, \dots, N-1$ ) is a complete orthonormal operator basis. For general  $N$  the pair  $U, V$  (of period  $N$ ) may be factorized into  $f$  mutually commutative pairs  $U_j, V_j$  of period  $\nu_j$ , corresponding to the decomposition of  $N$  into prime factors  $\nu_j$  ( $j = 1, \dots, f$ ). The different possible types of quantum degrees of freedom may now be classified according to the characteristics of the algebras generated by the pairs  $U(\nu), V(\nu)$  belonging to different primes  $\nu$ . The limiting case  $\nu \rightarrow \infty$  is best discussed in terms of the Hermitian operators  $q, p$  of which  $U, V$  are functions; in this way one obtains the familiar complementary pair of properties with continuous spectra.

P. W. Higgs (Edinburgh)

6447:

Phipps, T. E., Jr. Generalization of quantum mechanics. *Phys. Rev. (2)* **118** (1960), 1653-1658.

The paper proposes a generalization of quantum mechanics which preserves a correspondence to the Hamilton-Jacobi theory. The generalization is obtained by permitting  $S$  in the equations  $H(x, p, t)\psi = -\partial(S\psi)/\partial t$ ,  $p\psi = \partial(S\psi)/\partial x$  and  $-P\psi = \partial(S\psi)/\partial X$  to be a non-constant function. (For  $\psi = \text{const}$  one has the classical Hamilton-Jacobi theory and for  $S = \hbar/i$ , one has the Schrödinger equation.) A particular case, the relativistic theory of a particle moving in a Coulomb field, is worked out in detail. With a suitable ( $\psi$  dependent) choice of  $S$  it can be arranged that the equation has extra bound states. It is suggested that it may be possible to use these states to describe the states of nuclei.

A. S. Wightman (Princeton, N.J.)

6448:

Fierz, M.; Weisskopf, V. F. (Editors). *Theoretical physics in the twentieth century: A memorial volume to Wolfgang Pauli*. Interscience Publishers, New York-London, 1960. x + 328 pp. (1 plate) \$10.00.

This interesting volume contains twelve articles of varying length, written by eminent physicists of this our age of great discoveries. The articles fall, roughly speaking, into two categories: historical and personal reminiscences about the development of those fields of modern physics in which Pauli played an active role, and review articles concerned with the present status of various important branches of theoretical physics. In the first group we find the articles by Kronig, Heisenberg, Wentzel and van der Waerden about the development of quantum theory, Casimir's note on Pauli's attitude towards the theory of solid state, and Bargmann's account on the development of relativity. To the second category belong the review articles by Villars, Jost and Landau about modern developments concerning the quantum theory of fields, Peierls' paper on the quantum theory of solids, Fierz's critical paper on statistical mechanics and Wu's comprehensive review paper on neutrino physics. The distinction of these categories is of course not sharp and the topics dealt with in the various chapters are often overlapping. Because of the necessary limitation of available space the style of the articles is often rather condensed, but it is

beyond doubt that for the specialists this volume will prove to be not only a delightful reading material but also a useful reference work for quick orientation. Extensive bibliographical notes enhance also its usefulness.

*P. Roman (Boston, Mass.)*

6449:

**Bohr, Niels. Foreword.** Theoretical physics in the twentieth century (Pauli memorial volume), pp. 1-4. Interscience, New York, 1960.

6450:

**Kronig, R. The turning point.** Theoretical physics in the twentieth century (Pauli memorial volume), pp. 5-39. Interscience, New York, 1960.

6451:

**Heisenberg, W. Erinnerungen an die Zeit der Entwicklung der Quantenmechanik.** Theoretical physics in the twentieth century (Pauli memorial volume), pp. 40-47. Interscience, New York, 1960.

6452:

**Wentzel, Gregor. Quantum theory of fields (until 1947).** Theoretical physics in the twentieth century (Pauli memorial volume), pp. 48-77. Interscience, New York, 1960.

6453:

**Villars, F. Regularization and non-singular interactions in quantum field theory.** Theoretical physics in the twentieth century (Pauli memorial volume), pp. 78-106. Interscience, New York, 1960.

6454:

**Jost, Res. Das Pauli-Prinzip und die Lorentz-Gruppe.** Theoretical physics in the twentieth century (Pauli memorial volume), pp. 107-136. Interscience, New York, 1960.

Besides a comprehensive historical review of earlier developments concerning the covariant formulation of commutation rules for quantized fields, the connection between spin and statistics, and the CTP theorem, this paper discusses the following modern developments in considerable detail: The real and the complex Lorentz groups and their representations; the general theory of vacuum expectation values and in particular the theorem of Bargmann, Hall and Wightman, including its proof; strong and weak locality; the general proof of the CTP theorem and of the connection between spin and statistics. Besides the topics actually discussed, the article also gives a general idea about the new and potentially powerful methods of abstract field theory.

*P. Roman (Boston, Mass.)*

6455:

**Casimir, H. B. G. Pauli and the theory of the solid state.** Theoretical physics in the twentieth century (Pauli memorial volume), pp. 137-139. Interscience, New York, 1960.

6456:

**Peierls, R. E. Quantum theory of solids.** Theoretical physics in the twentieth century (Pauli memorial volume), pp. 140-160. Interscience, New York, 1960.

6457:

**Fierz, Markus. Statistische Mechanik.** Theoretical physics in the twentieth century (Pauli memorial volume), pp. 161-186. Interscience, New York, 1960.

6458:

**Bargmann, V. Relativity.** Theoretical physics in the twentieth century (Pauli memorial volume), pp. 187-198. Interscience, New York, 1960.

6459:

**van der Waerden, B. L. Exclusion principle and spin.** Theoretical physics in the twentieth century (Pauli memorial volume), pp. 199-244. Interscience, New York, 1960.

6460:

**Landau, L. D. Fundamental problems.** Theoretical physics in the twentieth century (Pauli memorial volume), pp. 245-248. Interscience, New York, 1960.

6461:

**Wu, C. S. The neutrino.** Theoretical physics in the twentieth century (Pauli memorial volume), pp. 249-303. Interscience, New York, 1960.

6462:

**Enz, Charles P. Bibliography Wolfgang Pauli.** Theoretical physics in the twentieth century (Pauli memorial volume), pp. 304-311. Interscience, New York, 1960.

6463:

**de Broglie, Louis. ★Éléments de théorie des quanta et de mécanique ondulatoire. Traité de Physique Théorique et de Physique Mathématique, III.** Gauthier-Villars, Paris, 1959. viii + 302 pp. 30.00 NF.

This is a remarkably clear introduction to wave mechanics, written by one of the pioneers of this field. The first-half of the book is devoted to a survey of the Maxwell electromagnetic theory, special relativity, classical statistical mechanics, black body radiation, and the Bohr-Sommerfeld quantum theory. The basic ideas of wave mechanics are then introduced by means of a thorough discussion of the correspondence principle and the Hamilton-Jacobi equation. The second part of the book is devoted mostly to interpretation problems, and only very little space is allotted to mathematical techniques. Exercises are totally lacking. This is rather unfortunate, since in general students can grasp new ideas only by working extensively with them. *A. Peres (Haifa)*

6464:

Beaufays, O. Détermination algébrique d'un spineur par un pseudo-vecteur. Acad. Roy. Belg. Bull. Cl. Sci. (5) 45 (1959), 859-869.

To every spinor  $\psi$  one can make correspond a pseudo-vector  $t_\mu$  by

$$t_\mu = (\bar{\psi}\psi)(\bar{\gamma}_\mu\gamma_5\psi) + (\bar{\gamma}_\mu\psi\psi)(\bar{\gamma}_\mu\psi).$$

The author discusses the inverse problem: to find the spinor  $\psi$ , some information about  $t_\mu$  being given. The following cases are explicitly solved: (1)  $t_\mu$  given (2 solutions); (2)  $t_\mu$  given, but for a real positive factor (2 solutions); and (3)  $t_\mu$  given, but for a real factor (4 solutions).

E. M. Bruins (Amsterdam)

6465:

Shuler, Kurt E.; Zwanzig, Robert. Quantum-mechanical calculation of harmonic oscillator transition probabilities in a one-dimensional impulsive collision. J. Chem. Phys. 33 (1960), 1778-1784.

Author's summary: "Quantum-mechanical vibrational transition probabilities  $P_{i \rightarrow f}(\epsilon)$  for harmonic oscillators, undergoing impulsive hard sphere collisions along the line of centers with an incident atom with relative kinetic energy  $\epsilon$ , have been computed by a machine (IBM-704) solution of the relevant Schrödinger equation. Curves for  $P_{i \rightarrow f}(\epsilon)$  over a range of  $\epsilon$  are presented for initial ( $i$ ) and final ( $f$ ) vibrational oscillator states  $i, f = 0, 1, 2$ , and 3. It is shown that this model of an inelastic collision gives rise to appreciable vibrational transitions  $v(i) \rightarrow v(f)$  with  $|\Delta v| > 1$  (in addition to  $|\Delta v| = 1$ ) in contrast to the Landau-Teller-Herzfeld adiabatic, first-order perturbation treatment which permits only transitions with  $|\Delta v| = 1$ . This result is discussed in relation to the dissociation of diatomic molecules and to the adsorption of atoms on solids. Averaged transition probabilities  $\bar{P}_{i \rightarrow f}(T)$  are computed for an incident beam of particles with a Maxwellian velocity distribution. It is pointed out that such averaged transition probabilities may give a misleading impression of the efficiency of translational-vibrational energy transfer if the  $P_{i \rightarrow f}(\epsilon)$  show a resonance type of behavior, i.e., a strong dependence of  $P_{i \rightarrow f}(\epsilon)$  on  $\epsilon$  over a small interval of  $\epsilon$ ."

6466:

Levinger, J. S.; Razavy, M.; Rojo, O.; Webre, N. Perturbation theory applied to the nuclear many-body problem. Phys. Rev. (2) 119 (1960), 230-240.

Perturbation theory in various forms is applied to nuclear matter at its normal density. The main object is to study the rapidity of convergence of the series, and this is done by comparing the second-order term with the total potential energy. Ordinary perturbation theory for weak forces and an expansion suitable for use with hard-core potentials are studied. It is found that the second-order terms are usually quite large, of the order of 20 Mev per particle, and that the tensor force can make a particularly large contribution. D. J. Thouless (Birmingham)

6467:

Moser, Josip. Beitrag zur Quantentheorie der Zentralkräfte. Bull. Soc. Math. Phys. Macédoine 9 (1958), 14-20. (Serbo-Croatian. German summary)

1094

The author uses the factorization method for exact solution of the quantum mechanical problem of the particle in the field of central force. It is shown that this method can be used only in the cases when the potential is proportional to  $r^2$  (harmonic oscillator),  $r^{-1}$  (Coulomb central force) or when it is constant. The Schrödinger equation for the particle in the field of central force is separated in polar coordinates, and by means of substitution can be further reduced to the governed differential equation  $(d_{xx} + \lambda - V_l)\psi_l = 0$ , where  $V_l = f(x)$ ,  $l$  a constant whole integer parameter and  $B_l = u_l(x) - d_x$  the operator of the factorization method. The function  $u_l(x)$  must be so determined that the operator  $B_l$  applied to the wave function  $\psi_l$  gives the new wave function  $\psi_{l+1}$  which is the solution of the mentioned differential equation with parameter  $l+1$ .

D. P. Rašković (Belgrade)

6468:

Epstein, S. T. On the anisotropy of inertia. Nuovo Cimento (10) 16 (1960), 587-588.

6469:

Elagin, Yu. P. Polarization of nucleons scattered from nuclei with non-zero spin. Ž. Eksper. Teoret. Fiz. 38 (1960), 1870-1871 (Russian. English summary); translated as Soviet Physics. JETP 11, 1344-1345.

Author's summary: "The optical model is used for an analysis of the polarization of nucleons scattered from nuclei with non-zero spin. It is shown that, in general, such nuclei lead to an additional polarization as compared to nuclei with the same optical parameters and zero spin."

6470:

Kerimov, B. K.; Arutyunyan, V. M. Polarization of electrons in elastic scattering with account of the finite size of the nucleus. Ž. Eksper. Teoret. Fiz. 38 (1960), 1798-1802 (Russian. English summary); translated as Soviet Physics. JETP 11, 1294-1297.

Authors' summary: "The scattering phase shifts and polarization of elastically scattered electrons are computed with account of the finite size of the scattering center. An expression for azimuthal asymmetry in double scattering has been obtained, as well as a correction to the usual Mott formula due to the second and fourth charge-density moments."

6471:

Gammel, J. L.; Christian, R. S.; Thaler, R. M. Calculation of phenomenological nucleon-nucleon potentials. Phys. Rev. (2) 105 (1957), 311-319.

Authors' summary: "An attempt to find a phenomenological nucleon-nucleon potential is described. The class of charge and velocity independent potentials with central and tensor parts of Yukawa shape with a hard core is considered. The depths, ranges, and core radii of such potentials with general spin and parity dependence are adjusted to fit experimental data. No potential of this type is found which fits all of the data."

6472:

Fabre de la Ripelle, Michel. Relation entre les paramètres du potentiel singlet pair de Gammel et Thaler. C. R. Acad. Sci. Paris **250** (1960), 3958-3959.

It is shown that for singlet even parity potentials of the type

$$V(r) = +\infty, \quad r < a \text{ (hard core)} \\ = \frac{V \exp(-\mu r)}{\mu r}, \quad r > a \text{ (Yukawa),}$$

which fit the n-p or p-p low energy parameters,

$$(V/\mu^3) \exp(-\mu a) = \text{const.}$$

This result is verified for a list of such potentials given by Gammel, Christian and Thaler [see preceding review].

J. L. Gammel (Los Alamos, N.M.)

6473:

Rosenberg, Leonard; Spruch, Larry. Bounds on scattering phase shifts: static central potentials. Phys. Rev. (2) **120** (1960), 474-482.

Authors' summary: "It has recently been shown that rigorous upper bounds on scattering lengths can be obtained by adding to the Kohn variational expression certain integrals involving approximate wave functions for each of the negative-energy states. For potentials which vanish identically beyond a certain point, it is possible to extend the method to positive-energy scattering; one obtains upper bounds on  $(-k \cot \eta)^{-1}$ , when  $\eta$  is the phase shift. In addition to the negative-energy states one must now take into account a finite number of states with positive energies lying below the scattering energy. The states in this associated energy eigenvalue problem are defined by the imposition of certain boundary conditions on the wave functions. A second approach, involving an associated potential-strength eigenvalue problem, is also used. The second method includes the first as a special case and, more significantly, can be extended to scattering by compound systems. If some states are not accounted for, a bound on  $\cot \eta$  is not obtained; nevertheless it is still possible to obtain a rigorous lower bound on  $\eta$ . Upper bounds on  $\eta$  may also be obtained, but in a way which is probably not too useful for many-body scattering problems."

6474:

Lehman, Guy W.; Shapiro, Kenneth A. Approximate analytic approach to the classical scattering problem. Phys. Rev. (2) **120** (1960), 32-36.

From the authors' summary: "An approximate analytic approach to the problem of determining differential scattering cross sections for classical central-field repulsive forces is described."

6475:

Allcock, G. R. Topics in the theory of dispersion relations. Nuclear Phys. **14** (1959), 177-198.

Author's summary: "The mathematics and physics of the theory of dispersion relations are discussed, with particular reference to the quantized field description of the scattering by a fixed centre. Various known analytic

continuations of physical data are examined for mathematical stability, and the dispersion relation at fixed momentum transfer is proved by a simple method which is throughout susceptible to the operations of numerical analysis. It is suggested that such stability is a necessary feature of a satisfactory proof.

"The scattering problem is formulated entirely in terms of the fields at large but finite distances from the scatterer, and this leads to a direct and perspicuous treatment of the 'unphysical' region. The effect of finite violations of microscopic causality is expressed quantitatively through a new dispersion relation which, while still dealing only with accessible scattering data, embodies within itself a fundamental length."

K. Johnson (Cambridge, Mass.)

6476:

Khuri, N. N. Analyticity of the Schrödinger scattering amplitude and nonrelativistic dispersion relations. Phys. Rev. (2) **107** (1957), 1148-1156.

Author's summary: "The Fredholm theory of integral equations is used to give a rigorous proof of the analyticity and boundedness of the ordinary nonrelativistic scattering amplitude for a fixed momentum transfer. The results follow from ordinary quantum mechanics and certain conditions on the potentials. These conditions are stated explicitly, and the bound states are treated with rigor. It is shown that the amplitude vanishes in the limit of large momenta, and thus simple dispersion relations are derived. Finally, it is proved that the partial-wave expansion is convergent in the unphysical region, provided the potentials satisfy the same conditions as above."

6477:

Petrina, D. Ya. Dispersion relations in the diffraction problem. Ukrain. Mat. Ž. **10** (1958), 405-412. (Russian. English summary)

The results of Khuri [see above, #6476] on the analyticity of the scattering amplitude for the Schrödinger equation are extended to the equation of scattering by a finite body  $B$  of arbitrary shape, and it is shown that the scattering amplitude determines the Fourier transform of the characteristic function of  $B$ . L. Gårding (Lund)

6478:

Charap, J. M.; Fubini, S. P. The field theoretic definition of the nuclear potential. I. Nuovo Cimento (10) **14** (1959), 540-559. (Italian summary)

This paper discusses the construction of an energy independent two-nucleon potential which will reproduce the field-theoretic scattering amplitude in some energy range. Nucleons and mesons are taken to be scalar particles, a more realistic treatment being promised for a future paper.

The paper consists of two parts. In the first, the nuclear scattering amplitude  $G$  is calculated by perturbation theory in the  $(1+2)$  meson approximation, the corresponding potential being constructed from the equation  $G = V + V(E-H)^{-1}G$  by successive approximations. The resulting potential turns out to be independent of energy in the adiabatic limit  $\eta^2 \ll M^2$ , where  $2\eta$  is the relative momentum in the  $CM$  frame, and  $M$  is the mass of a nucleon ( $\hbar = c = 1$ ).

In the second part of the paper, the authors remark that Khuri's dispersion relation between a potential  $V$  and its scattering amplitude [see above, #6476] has the same form as a certain dispersion relation for  $G$  when absorptive terms of order  $(\eta^2/\mu M)$  are neglected ( $\mu$  = meson mass).  $V$  is then constructed from  $G$  by representing  $G$  at  $\eta^2=0$  by a dispersion relation in which mesons are the only sort of intermediate particle, and by using unitarity.

The significance of the static limit  $\mu/M \rightarrow 0$  is discussed.

A. Herzenberg (Manchester)

6479:

Klein, Abraham. Mandelstam representation for potential scattering. *J. Mathematical Phys.* **1** (1960), 41-47.

This paper is concerned with a derivation of a representation for the scattering amplitude in non-relativistic quantum theory which expresses the analyticity in the variables  $s$  (energy) and  $t$  (momentum transfer). It is the potential scattering analogue of the Mandelstam conjecture in field theory [S. Mandelstam, *Phys. Rev.* (2) **112** (1958), 1344-1360; MR **20** #5057]. Attention is confined to scattering by a Yukawa potential. The analyticity of the scattering amplitude as a function of  $t$  for fixed real  $s$  is established by a direct examination of the three dimensional Born series for the amplitude. Aside from the possibility of an essential singularity at infinity (which is disposed of by a detailed argument based on knowledge of partial wave amplitude analyticity proved by other means) one can then conclude, by using the known single variable dispersion relation [N. N. Khuri, review #6476] that the scattering amplitude is analytic in the topological product of the cut planes of  $s, t$ . Actually the author's derivation is not complete because the requisite analyticity in  $t$  is proved only for finite values of  $s$ , whereas the single variable dispersion relations involve values of  $s$  ranging from zero to infinity.

M. L. Goldberger (Princeton, N.J.)

6480:

Chang, T. S. A simple proof of dispersive relations. *Sci. Sinica* **9** (1960), 459-465.

The author attempts to derive dispersion relations by considering the scattering amplitude for imaginary meson mass  $\sqrt{\xi}$ :

$$M(\Delta, \omega, \xi) \propto \int d^4x \exp i(\omega x_0 - \epsilon \cdot x(\omega^2 - \xi - \Delta^2)^{1/2})$$

$$\langle p' | \theta(x) \left[ j\left(\frac{x}{2}\right), j\left(\frac{-x}{2}\right) \right] | p \rangle.$$

By expanding the commutator over intermediate states of mass  $M_n$  it is shown that  $M(\Delta, \omega, \xi)$  has the form

$$M(\Delta, \omega, \xi) = \sum_n \left[ \frac{N_1((\omega^2 - \Delta^2 - \xi)^{1/2}, M_n)}{\omega - E_p - (M_n^2 + \omega^2 - \xi - \Delta^2)^{1/2}} + \frac{N_2((\omega^2 - \Delta^2 - \xi)^{1/2}, M_n)}{\omega - E_p + (M_n^2 + \omega^2 - \xi - \Delta^2)^{1/2}} \right].$$

Now the branch point at  $M_n^2 + \omega^2 - \xi - \Delta^2 = 0$  is ignored. No reason for this is given. The sum (at  $\Delta^2 = -\xi$ ) is regular in the upper  $\omega$ -half plane, leading one to believe that  $N(\omega, M_n)$  has the same property, and hence that

$N((\omega^2 - \Delta^2 - \mu^2)^{1/2}, M_n)$  has too. Again ignoring the branch point, each term in the sum for  $M(\omega, \mu^2, \Delta^2)$  has the required analyticity properties.

The second part discusses analyticity of phase shifts in potential scattering, where  $V(r) \sim \exp(-\alpha r)$ . The author expands the wave function in powers of the potential; each term is shown to be regular in the strip  $|\operatorname{Im} k| < \frac{1}{2}\alpha$  [i.e., a result not as good as that of N. N. Khuri, #6476]. Here again no rigorous results are obtained.

R. F. Streater (Princeton, N.J.)

6481:

Haller, Kurt. Divergence-free iterative expansion of the  $S$  matrix in a field theory. *Phys. Rev.* (2) **120** (1960), 1045-1057.

It is suggested that the divergences in conventional quantum field theory might be due to an incorrect treatment of the asymptotic condition, which should be replaced by the formalism of Ekstein [*Phys. Rev.* (2) **101** (1956), 880-890; MR **17**, 809]. The author treats in detail the static model of scalar bosons interacting with a point source. Integral equations of the Chew-Low type are set up, and an iterative procedure in powers of the coupling constant is developed. It is found that only convergent integrals appear, and up to 6th order the answer is the same as that obtained by renormalization of the usual Feynman graphs. When the method is applied to a non-renormalizable theory, one obtains divergent integrals, but the divergence is less than in the conventional theory. The method is not applicable to theories with boson self-energies.

R. F. Streater (Princeton, N.J.)

6482:

Volkov, D. V.  $S$ -matrix in the generalized quantization method. *Z. Eksper. Teoret. Fiz.* **38** (1960), 518-523 (Russian. English summary); translated as *Soviet Physics. JETP* **11**, 375-378.

This is a contribution to the field theory of particles satisfying a statistics more general than Fermi or Bose statistics. There is some work on the evaluation of the  $S$ -matrix, which yields results similar to those already obtained by I. E. McCarthy [*Proc. Cambridge Philos. Soc.* **51** (1955), 131-140; MR **16**, 548]. This is followed by the explicit evaluation of the matrix element for the scattering of one particle by another, in lowest order perturbation theory.

H. S. Green (Adelaide)

6483:

Hara, Yasuo; Miyazawa, Hironari. Dispersion relations in nucleon-nucleon scattering. *Progr. Theoret. Phys.* **23** (1960), 942-956.

Authors' summary: "Two-pion contribution to the absorptive part of nucleon-nucleon scattering amplitudes in the unphysical region is calculated using the dispersion relations for pion-nucleon scattering. The dispersion relations with this absorptive part are used for analyzing nucleon-nucleon scattering data at low energy and at moderate energy, and we find good agreement if we choose the coupling constant as

$$\frac{f^2}{4\pi} = 0.08 \pm 0.01."$$

6484:

Kaus, Peter E.; Watson, W. K. R. Dispersion relations for Bloch functions. *Phys. Rev. (2)* **120** (1960), 44-48.

Authors' summary: "It is shown that the Floquet factor  $e^{ik(E)s}$  is analytic in the upper half complex energy plane, thus enabling a set of four dispersion relations to be derived from this expression as a direct result of the application of Cauchy's theorem. These relations are characterized by their ability to relate the wave number  $k$  at one energy to the wave number at all others. In particular, the imaginary part of the wave number  $k_i$  in the forbidden gap may be equated to an integral of a function of the real part of the wave number  $k_r$  over allowed energies. As an application of these dispersion relations a theorem regarding the location of the branch points has been established."

6485:

MacDowell, S. W. Analytic properties of partial amplitudes in meson-nucleon scattering. *Phys. Rev. (2)* **116** (1959), 774-778.

In order to obtain dispersion relations for the partial wave amplitudes in meson-nucleon scattering, the Mandelstam representations for the covariant scattering amplitudes [S. Mandelstam, *Phys. Rev. (2)* **112** (1958), 1344-1360; MR **20** #5057] is employed. Instead of the amplitudes  $f_l^\pm$  for transitions in given states of total angular momentum  $j = l \pm \frac{1}{2}$ , the combinations  $\phi_l^+ = w^{-1}(f_l^+ + f_{l+1}^-)$ ,  $\phi_l^- = (f_l^+ - f_{l+1}^-)$  are found to be more convenient. The analytic properties of these amplitudes are discussed on the basis of the Mandelstam representation, and they are shown to be analytic functions of the energy throughout the complex plane except for cuts along the real axis and a cut along a circle with its centre on the negative real axis. With this information dispersion relations are set up, but owing to the appearance of non-physical regions, the unitarity condition is not simple in general, and approximations to the integrals are required. (Similar considerations have been given by W. R. Fraser and J. R. Fulco [see review below].) C. A. Hurst (Adelaide)

6486:

Frazer, William R.; Fulco, Jose R. Partial-wave dispersion relations for pion-nucleon scattering. *Phys. Rev. (2)* **119** (1960), 1420-1426.

Authors' summary: "Partial-wave dispersion relations for pion-nucleon scattering are derived from the Mandelstam representation. The symmetry of the representation is used to obtain expressions for the discontinuities across the unphysical branch cuts."

6487:

Noyes, H. Pierre. Energy dependence of the nucleon-nucleon phase shifts. *Phys. Rev. (2)* **119** (1960), 1736-1742.

From the author's summary: "Starting from the analytic structure of partial wave amplitudes predicted by the Mandelstam representation, relativistic formulas are derived for the energy dependence of the phase shifts for nucleon-nucleon scattering, neglecting inelastic processes. These formulas depend on integrals over functions defined by a (numerically) soluble integral equation whose

kernel is determined from the absorptive part of the amplitude in the nonphysical region. The contribution to this kernel from single-pion exchange is explicitly exhibited and the contribution from two-pion exchange is calculable. A generalization of the formulas to include phenomenological constants representing the unknown contribution of multimeson and other particle exchanges is given."

6488:

Tsai, Yung Su. High-energy electron-electron scattering. *Phys. Rev. (2)* **120** (1960), 269-286.

Author's summary: "The radiative corrections to the electron-electron scattering to order  $\alpha^3$  are calculated for (1) the colliding beam experiment and (2) the experiment in which the target electron is at rest initially. The contributions from high-energy real photons are included. The two-photon exchange diagrams are found to give only negligible contributions to the cross sections after infrared cancellation. The effect due to the possible breakdown of quantum electrodynamics is discussed. A preliminary study on the electron-positron colliding beam experiment involving various interactions is made. The vacuum polarizations involving heavier particles than an electron pair in the closed loop are investigated."

6489:

Swan, P. An improved approximation for scattering problems. *Nuclear Phys.* **18** (1960), 245-270.

A modified Born approximation is obtained from the exact expression

$$\sin \delta_l = -k^{-1} \int_0^\infty j_l(kr) U(r) u_l(r) dr$$

by substituting for the wave function  $u_l(r)$  the approximation

$$u_l(r) \approx j_l(kr) \cos \delta_l - f_l(r) n_l(kr) \sin \delta_l,$$

where  $f_l(r)$  is a "form factor"

$$f_l(r) = 1, \quad r > \text{some cutoff radius,} \\ = 0, \quad r = 0.$$

The method is modified to be applicable to non-local Schrödinger equations of the type encountered when the no distortion approximation is applied to the three body scattering problem (for example).

Extensive numerical comparisons with exact calculations are included. J. L. Gammel (Los Alamos, N.M.)

6490:

Newton, Roger G. Analytic properties of radial wave functions. *J. Mathematical Phys.* **1** (1960), 319-347; errata, 452.

A review article of the Jost-Bargmann work and related aspects of scattering theory in non-relativistic quantum mechanics. J. M. Blatt (Murray Hill, N.J.)

6491:

Borchers, H.-J. Über die Mannigfaltigkeit der interpolierenden Felder zu einer kausalen S-Matrix. *Nuovo Cimento* (10) **15** (1960), 784-794. (English and Italian summaries)

Assuming the existence of a non-trivial causal  $S$ -matrix, the author shows that the local fields corresponding to a causal  $S$ -matrix may be partitioned into equivalence classes such that the fields in any one class mutually commute. Assuming that a given class contains one field, the author constructs other fields belonging to the same class (for his particular example, in number equal to the cardinality of the continuum). He is unable to decide whether or not there could be more than one class associated with a given  $S$ -matrix.

A. J. Coleman (Kingston, Ont.)

6492:

Bogolyubov, N. N.; Vladimirov, V. S. On some mathematical problems of quantum field theory. Proc. Internat. Congress Math. 1958, pp. 19-32. Cambridge Univ. Press, New York, 1960.

This address is a clear exposition of some problems of the theory of distributions and analytic functions which have arisen in quantum field theory, especially in dispersion theory. The subjects treated include the multiplication of distributions, and the edge of the wedge theorem and its generalizations.

A. S. Wightman (Princeton, N.J.)

6493:

Streater, R. F. Special methods of analytic completion in field theory. Proc. Roy. Soc. London. Ser. A 256 (1960), 39-52.

The author studies some problems concerning distributions and analytic functions of several complex variables obtained following Wightman [Phys. Rev. (2) 101 (1956), 860-866; MR 18, 781] from vacuum expectation values which occur in quantum field theory. The main tool is the Jost-Lehmann-Dyson representation [Dyson, *ibid.* (2) 110 (1958), 1460-1464; MR 20 #1537] of the Fourier transform of the most general function which vanishes for space-like value of its argument. Among the results given are a brief proof of the edge of the wedge theorem, a representation {which is not unique} of the most general vacuum expectation value of a double commutator,  $D = \langle 0 | [A(x_1), [B(x_2), C(x_3)]] | 0 \rangle$ , which includes the conditions due to locality and positive energy spectrum, but not those due to the Jacobi identity or mass thresholds, and representations for the most general functions

$$\langle 0 | A(x_1) \cdots B(x_{n-2}) [C(x_{n-1}), E(x_n)] | 0 \rangle,$$

$$\langle 0 | [A(x_1), B(x_2)] [C(x_3), E(x_4)] | 0 \rangle,$$

which include the conditions of positive energy spectrum and locality in the commutators. Concerning  $D$ , the author quotes an unpublished result of Symanzik that Lorentz invariance, positive energy spectrum, C.T.P. theorem and locality of the inner commutator,  $D=0$  for  $x_2-x_3$  space-like, imply locality of the outer commutator,  $D=0$  for  $x_1-x_2$  and  $x_1-x_3$  space-like. Using these representations, the author finds the envelope of holomorphy of the union of the domains of regularity of some of the associated Wightman functions; in particular he simplifies part of the Källén-Wightman calculation [Källén and Wightman, Mat.-Fys. Skr. Danske Vid. Selsk. 1 (1958), no. 6, 1-58; MR 22 #3470] of the envelope of holomorphy of the threefold vacuum expectation value.

O. W. Greenberg (Cambridge, Mass.)

6494:

Streater, R. F. Some integral representations in field theory. Nuovo Cimento (10) 15 (1960), 937-948. (Italian summary)

The author generalizes the representation of the vacuum expectation value of the double commutator given in the paper reviewed above to the triple commutator,

$$T = \langle 0 | [A(x_1), [B(x_2), [C(x_3), D(x_4)]]] | 0 \rangle,$$

and finds a representation of  $T$  which includes positive energy spectrum, C.P.T. theorem, and the locality of the inner and middle commutators,  $T=0$  if  $x_3-x_4$  is space-like, or if  $x_2-x_3$  and  $x_2-x_4$  are space-like, but not the locality of the outer commutator,  $T=0$  if  $x_1-x_2$ ,  $x_1-x_3$  and  $x_1-x_4$  are space-like. The representation for  $T$  is not shown to be the most general one. Similar partial results are found for the vacuum expectation value of the  $n$ -fold nested commutator, and for some other vacuum expectation values. Using these representations the author makes conjectures about the envelope of holomorphy of the associated Wightman functions.

O. W. Greenberg (Cambridge, Mass.)

6495:

Streater, R. F. Double commutator in quantum field theory. J. Mathematical Phys. 1 (1960), 231-233; errata, 452.

The author inserts mass threshold conditions into the representation of the vacuum expectation value of the double commutator which he gave in the paper reviewed above. He gives inequalities which these mass thresholds must satisfy. The possibility of a discrete mass intermediate state is not included.

O. W. Greenberg (Cambridge, Mass.)

6496:

Sokolov, A. A.; Kolesnikova, M. M. On the scattering of transversely polarized fermions. Ž. Eksper. Teoret. Fiz. 38 (1960), 1778-1785 (Russian. English summary); translated as Soviet Physics. JETP 11, 1281-1285.

Authors' summary: "The elastic scattering of transversely polarized fermions is considered. It is shown that in the ultrarelativistic case scattering of the longitudinally polarized fermions is characterized by the four dimensional intrinsic angular momentum tensor. In the latter case the scattered fermions remain transversely polarized in interactions proportional to the matrices  $\rho_1$  and  $\rho_2$ ; on the contrary, in interactions proportional to the matrices  $\rho_3$  and  $\rho_4$ , the longitudinal component may appear after scattering (if the particle has a non-zero rest mass)."

6497:

Siegert, A. J. F. Field operators for bosons with impenetrable cores. I. Equations which replace the commutation rules. Phys. Rev. (2) 116 (1959), 1057-1062.

To take account of the hard cores of a set of Bose particles, the author proposes to modify the commutation relations of the (second-quantized) field operators. For hard spheres, the usual commutation relations are replaced in the spherical region  $s_x$  about  $x$ ,  $|x-x'| \leq a$ , by the requirements:  $\psi(x)\psi(x') = \psi'(x)\psi'(x') = 0$ , and  $\psi(x)\psi'(x') = \delta(x-x')[I + \sum (\lambda)^l \mathcal{N}_l(s_x)]$ , where  $\mathcal{N}_l(s_x)$ ,  $l=1, 2, 3, \dots$ .

$N(s_x)$ , are the operators for the number of particles, pairs, triplets, ..., etc., in the domain  $s_x$ . The number operator in a finite domain  $S$  is bounded by the maximum number  $N(S)$  of hard spheres which may be packed in  $S$ . All of the various operator equations are verified by appealing to the configuration-space representation. It is proposed to treat interacting hard-core bosons as an interaction among quanta of the modified field operators, and thus to avoid an infinite potential in the Hamiltonian, or to circumvent the use of the scattering amplitude in the method of Bogoliubov [Acad. Sci. USSR. J. Phys. 11 (1947), 23-32; MR 9, 168].

J. R. Klauder (Murray Hill, N.J.)

6498:

Rohrlich, F. *The classical electron*. Lectures in theoretical physics (Boulder, Colo., 1959), pp. 240-268. Interscience, New York, 1960.

A modern survey of various important questions related to the classical theory of the electron. The treatment is consistently covariant throughout. It starts with an elegant summary of the basic equations of classical electrodynamics, including the various types of solutions and the equation of motion. Then a precise definition of the radiation field is given and the field reaction and radiation reaction are discussed, emphasizing the difference between these two concepts. Dirac's 1938 formulation of electrodynamics and his comments on the equations of motion are summarized and critically analyzed. The self-energy and self-stress problem is discussed in modern terms, i.e., renormalization concepts are used. As an application, the uniformly accelerated motion of a charged particle is worked out. In conclusion, it is proposed that Schott's acceleration-energy should be taken seriously.

P. Roman (Boston, Mass.)

6499:

Shimazu, Haruo. *On the non-local boundary condition in quantum field theory*. Progr. Theoret. Phys. 23 (1960), 821-828.

The author attempts to set up a Hamiltonian formalism corresponding to the indefinite metric  $S$ -matrix theory suggested by Bogolyubov, Medvedev and Polivanov [*Voprosy teorii dispersionnykh sootnoshenii*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958; MR 21 #1150]. The difficulty of doing this is illustrated by the fact that the resulting Hamiltonian is not Hermitian.

J. C. Polkinghorne (Cambridge, England)

6500:

Terletsky, J. P. *Sur la théorie statistique des champs non linéaires*. J. Phys. Radium 21 (1960), 771-775. (English summary)

Author's summary: "The canonical and microcanonical distributions of probabilities are introduced for an arbitrary classical, non-linear field. The expressions of the functional density of probability of the lines of universe are found, which give the particle-like solutions, of the non-linear field equations. The Feynman's formulation of quantum mechanics is only a special case of a general canonical distribution, with an imaginary temperature. That temperature is interpreted as the statistical temperature of a thermostat constituted of particles having imaginary masses. The Planck constant is then a factor proportional to the temperature of the 'imaginary' thermostat."

6501:

Ozaki, Shoji. *Selection rules for interaction types in quantum field theory*. Progr. Theoret. Phys. 23 (1960), 221-228.

The author factors the Klein-Gordon differential operator into two different first order differential operators which involve  $\gamma_5$  in addition to the usual  $\gamma_\mu$  matrices. In this way he gets a "generalized Dirac equation" which differs from the usual one by  $\exp a\gamma_5$ , where  $a$  is a number, in one term. For the source-free equation, he removes this  $\exp a\gamma_5$  factor by a  $\gamma_5$  dependent similarity transformation. The author now proposes an ad hoc requirement for an interaction: the  $\gamma_5$  dependent similarity transformation must transform the generalized Dirac equation with a given source into the ordinary Dirac equation with the same source. The author examines various interactions from this viewpoint, and finds, for example, that among the Fermi interactions only vector and axial vector satisfy his requirement.

O. W. Greenberg (Cambridge, Mass.)

6502:

Nickle, H.; Serber, R. *Charged-scalar strong-coupling theory*. Phys. Rev. (2) 119 (1960), 449-457.

Authors' summary: "A treatment of the charged-scalar strong-coupling theory is given which employs a somewhat different choice of variables than that usually used; one which is more convenient for a discussion of the effects of quantum mechanical field fluctuations. The expansion parameter of the strong-coupling theory is shown to be  $(1/g^2) \ln(1/Ka)$ , where  $a$  is the source radius. The isobar energy is calculated to order  $1/g^4$ , and terms of order  $(1/g^2) \ln(1/Ka)$  relative to the leading  $1/g^2$  term are found to appear. Similar terms occur in the charge-renormalization factor. The logarithmic term in the isobar energy is found to be precisely that required to renormalize the charge; that is, the isobar energy, if expressed in terms of the renormalized coupling constant, is explicitly independent of the source radius."

6503:

Kawakami, Ichiro. *On ratios of the renormalized and the bare coupling constants in the beta decay interaction*. Progr. Theoret. Phys. 24 (1960), 27-34.

Author's summary: "The ratio of the renormalized and the bare coupling constants in the  $\beta$ -decay interaction is calculated in two ways. The first is evaluation of the contribution from the process in which a neutron and an antiproton, which virtually exchange an infinite number of pions, are annihilated into an electron-neutrino (or antineutrino) pair. For this purpose a dispersion relation for the vertex function of the  $\beta$ -decay is assumed. Using this relation the ratio is calculated in the 'ladder' approximation of Federbush, Goldberger and Treiman. The result is that the ratio is smaller than unity. The second way is calculating the ratio so as to include the main contribution from virtual nucleon pairs. For this purpose we derive an equation for the vertex operator of the  $\beta$ -decay. The vertex operator is calculated using Goto and Machida's approximation. The result is that the ratio for the axial vector coupling constants is larger than unity, while that for the vector coupling constants is smaller than unity. The experimental value for the ratio ( $\sim 1.2$ ) is reproduced at a rather small cutoff energy ( $\sim$  one-pion rest mass)."

6504:

Segal, I. E. Quasi-finiteness of the interaction hamiltonian of certain quantum fields. *Ann. of Math.* (2) **72** (1960), 594-602.

Let  $A_1, A_2, \dots$  and  $B_1, B_2, \dots$  be two sequences of self-adjoint operators in a Hilbert space  $K$ , all of which commute. The author's main result is that there exists a Hilbert space  $L$  and sequences  $Q_1, Q_2, \dots$  and  $P_1, P_2, \dots$  of self-adjoint operators on the product  $K \times L = M$  of these Hilbert spaces such that the  $Q_i$  and  $P_j$  satisfy the Heisenberg commutation relations (in Weyl form) and such that  $\sum_n [(A_n \times I)P_n + (B_n \times I)Q_n]$  has a closure which is essentially self-adjoint.

Sums of the indicated form occur in quantum field theory but do not yield well defined operators when one uses the standard solution of the Heisenberg commutation relations as the set of  $P$ 's and  $Q$ 's. Thus one can eliminate certain "divergences" by simply changing from one solution of the Heisenberg commutation relations to another. The author submits this fact as evidence that the divergences are of analytical rather than physical origin and suggests (as he has done in more detail elsewhere [*Mat.-Fys. Medd. Danske Vid. Selsk.* **31** (1959), no. 12; MR **22** #3477]) that one seeks to define the Hamiltonian in terms of automorphisms of the algebra of bounded observables rather than directly as a self-adjoint operator.

G. W. Mackey (Cambridge, Mass.)

6505:

Zumino, Bruno. Gauge properties of propagators in quantum electrodynamics. *J. Mathematical Phys.* **1** (1960), 1-7.

In quantum electrodynamics the gauge invariance of the field equations has led to much formal confusion. Only the transverse electromagnetic field is kinematically independent, the longitudinal field is given as a function of the charge degrees of freedom. This, of course, is a physically meaningful statement but obscures the covariance of the field equations. Numerous tricks to avoid the separation of the field into its component parts have been proposed over the years, all of which are physically artificial. In this note the author shows [as did Schwinger and the reviewer several years ago, unpublished] how to combine the best of both worlds, by making use of the gauge properties of the equations for Green's functions rather than of the equations for the field operators. In the equations for Green's function, one has complete freedom of choice of "quantum gauge" whereas in the operator equations one is restricted to work with the so-called "Coulomb" gauge if one restricts oneself to an operator formalism that works with only physically meaningful quantities. The relation between the various formalisms is established in this note by showing how Green's functions in the various "quantum" gauges are related.

K. Johnson (Cambridge, Mass.)

6506:

Okubo, S. The gauge properties of Green's functions in quantum electrodynamics. *Nuovo Cimento* (10) **15** (1960), 949-958. (Italian summary)

The relation between Green's functions of certain special classes of gauges is derived using the combinatorial methods of Caianiello. The results were previously published by B. Zumino [see preceding review].

K. Johnson (Cambridge, Mass.)

6507:

Redmond, P. J. Some consequences for quantum electrodynamics of an essential singularity at  $\alpha=0$ . *Nuovo Cimento* (10) **14** (1959), 771-777. (Italian summary)

6508:

Johnson, Kenneth. Quantum electrodynamics in the infinite energy limit. *Ann. Physics* **10** (1960), 536-552.

Previous discussions of the convergence properties of quantized field theories in general and quantum electrodynamics in particular have, admittedly, been simple-minded in the respect that one freely allows oneself to interchange orders of various limiting processes and integrations as is customary in theoretical physics. The author of the present paper states in his introduction that he wants to reexamine quantum electrodynamics from a somewhat more strict mathematical point of view and, in particular, try to avoid such interchanges as much as possible. The author thinks that the best formulation of electrodynamics is obtained in the so-called radiation gauge. This formulation has the disadvantage that it is not explicitly covariant and, consequently, that spectral representations, etc., are more complicated than in the more conventional Lorentz gauge. Consequently, the formal difficulties increase considerably as compared to previous investigations. In particular, the author has an extra dependence of the spectral weight for the two point function (the photon propagator) on a new momentum variable. This means that he introduces a new parameter in the limiting processes which has not been present before. This new parameter leads to new difficulties with uniform convergence and the author suggests that these new difficulties could possibly save electrodynamics.

G. Källén (Lund)

6509:

Salam, Abdus. An equivalence theorem for partially gauge-invariant vector meson interactions. *Nuclear Phys.* **18** (1960), 681-690.

Author's summary: "Consider a massive neutral vector meson  $U_\mu$  interacting with other spin zero or spin  $\frac{1}{2}$  particles  $\phi$ . If the interaction Lagrangian consists of two parts  $L_1(\phi, U) + L_2(\phi)$  where  $L_1(\phi, U)$  is (together with kinetic energy terms) invariant for the gauge transformations  $\phi = \exp(iQ\Lambda)\phi'$ ,  $U_\mu = A_\mu - \kappa^{-1}(\partial\Lambda/\partial x_\mu)$  and  $L_2$  is not gauge-invariant, it is shown that the conventional  $S$ -matrix  $T \exp i(\int L_1(\phi, U) + L_2(\phi))$  identically equals  $T \exp i(\int L_1(\phi, A) + L_2(\phi'))$ . Since all  $\Lambda$ -dependent effects are now contained and displayed explicitly in  $L_2(\phi')$  it is possible to give a precise discussion of the renormalization problem for vector meson theories. This is because the suspected non-renormalizability of these theories arises from  $\partial\Lambda/\partial x_\mu$  terms in the conventional  $S$ -matrix."

P. T. Matthews (London)

6510:

Kamefuchi, S. On Salam's equivalence theorem in vector meson theory. *Nuclear Phys.* **18** (1960), 691-696.

Author's summary: "A simple proof is given of the equivalence theorem discussed in the preceding paper [6509]. Some comments are also made on renormalizability of vector meson fields." P. T. Matthews (London)

6511:

Kalitkin, N. N. The Thomas-Fermi model of the atom with quantum and exchange corrections. *Ž. Èksper. Teoret. Fiz.* **38** (1960), 1534-1540 (Russian. English summary); translated as Soviet Physics. JETP **11**, 1106-1110.

Following the work by Kiržnic [same *Ž.* **32** (1957), 115-123; **35** (1958), 1545-1557; MR **19**, 1130; **21** #5354], quantum and exchange correction to the energy of the Thomas-Fermi atom are derived. The expression for the thermodynamical quantities and the first order corrections to them for the finite cold atom are given. The extrapolation of the model into the region of normal density is considered. The numerical results show that the present treatment reproduces the experimental curves for the pressure and energy better than the Thomas-Fermi, the Thomas-Fermi-Dirac and the Weizäcker models, and the gross feature of the experimental dependence of the density of the uncompressed matter on the charge of the nucleus better than the infinite cold atom.

T. Sasakawa (Cambridge, Mass.)

6512:

Møller, Nils Henrik. Calculation of atomic energy levels for electronic configurations with one electron outside a nearly closed shell and two holes in this shell for LS and pair couplings. *Ark. Fys.* **18**, 135-157 (1960).

Formulae are first derived for obtaining the energy matrix for configurations of 2 or 3 electrons outside closed shells. Similar formulae are then given for the case of two holes in a nearly closed shell and one outer electron. Complete tables are shown for configurations of the type  $p^4$ , in both Russel-Saunders coupling and pair coupling.

D. F. Mayers (Oxford)

6513:

Dalgarno, A.; Stewart, A. L. The screening approximation for the helium sequence. *Proc. Roy. Soc. London. Ser. A* **257** (1960), 534-540.

Authors' summary: "Perturbation methods are employed to calculate a wide range of atomic parameters relating to the ground states of the helium iso-electronic sequence, the unperturbed eigenfunctions being products of screened hydrogenic orbitals. The screening constant enters as a disposable parameter. A means of choosing it is described which yields different values for different atomic properties. It is shown quantitatively that the simple calculations involved thereby lead to results of high accuracy."

6514:

Derrick, G. H. On the choice of D-state wave functions in triton calculations. *Nuclear Phys.* **18** (1960), 303-309.

Author's summary: "A set of D-state wave functions with the correct asymptotic properties is suggested for use in variation calculations of the triton energy. The matrix elements of the kinetic and potential energy operators are evaluated for these functions."

6515:

Hayakawa, Satio; Marumori, Toshio. A remark on the moments of inertia of rotating nuclei. *Progr. Theoret. Phys.* **18** (1957), 396-404.

Authors' summary: "The quantum mechanical description of a two dimensional rotating system of particles is studied for the purpose of elucidating the meaning of the effective moments of inertia of nuclei. We employ the method of canonical transformations and obtain the rotational kinetic energy of a 'canonical form' by introducing an internal angular momentum  $L_{in}$ . The Coriolis force arising from the coupling of  $L_{in}$  with the rotational angular velocity is found to be responsible for the deviation of the moment of inertia from its hydrodynamical value. The explicit form of  $L_{in}$  is given as a function of the positions and momenta of individual particles. This form of  $L_{in}$  should be used for deriving the effective moment of inertia. Comparison with the cranking model of Inglis is discussed."

6516:

Suh, Kiu S. Momentum space representation and photodisintegration of the deuteron. *Amer. J. Phys.* **28** (1960), 327-332.

Author's summary: "The momentum space representation is discussed in general and an application is made, deriving the cross section of the photodisintegration of the deuteron, using the Hulthen wave function. The theoretical results are compared with the experimental values, and the Hulthen wave function parameter is determined."

6517:

Demeur, M.; Janssens, P.; Lardinois, J. États individuels au seuil de fission. *Nuclear Phys.* **18** (1960), 280-285. (English summary)

Authors' summary: "A dumb-bell potential has been quantized in order to find properties of individual levels in a nucleus near the fission threshold."

6518:

Arima, Akito. Spin-orbit splitting and tensor force. *Nuclear Phys.* **18** (1960), 196-223.

Author's summary: "The second-order effect of the tensor force is calculated in nuclei which have several nucleons outside closed shells."

6519:

Mitra, A. N.; Narasimham, V. L. Role of spin-orbit force in nuclear interactions. *Nuclear Phys.* **14** (1959/60), 407-428.

Authors' summary: "The two-body problem (scattering and bound state) has been solved exactly by considering an extension of Yamaguchi's non-local separable potential so as to include a two-body spin-orbit force. The parameters of the potential have been determined by obtaining their exact relationships with the low-energy experimental data (scattering length, effective range, deuteron binding energy, quadrupole and magnetic moments and the D-state probability). It is found that in order to fit the data, particularly the magnetic moment, the strength of the spin-orbit force must be comparable to that of the tensor force."

O. Hara (Minneapolis, Minn.)

6520:

Monsonogo, Georges. *Modèle unifié pour les phénomènes de photodésintégration nucléaire vers 20 MeV*. J. Phys. Radium **21** (1960), 765-770. (English summary)

Author's summary: "We formulate a unified model for the study of the giant resonance, which takes into account the individual and collective character of this phenomenon. We perform two unitary transformations on the Hamiltonian of the nucleus, introducing two new conjugate operators  $\alpha$  and  $\pi$ , with a subsidiary condition on the eigenfunctions since we have increased the number of degrees of freedom. In the new Hamiltonian, there are: a collective part in terms of  $\alpha$  and  $\pi$ , an intrinsic part and interaction terms. This treatment based on the success of the Bohr and Mottelson phenomenological model seems to be successful: the theoretical calculations of the energy, width and cross section of the giant resonance, agree with the experimental data. The subsidiary condition plays an important rôle in the calculation of the width. We study also the effect of neutron-proton exchange forces and the angular distribution of the emitted particles."

6521:

Baker, George A., Jr.; McCarthy, Ian E.; Porter, Charles E. Application of the phase space quasi-probability distribution to the nuclear shell model. Phys. Rev. (2) **120** (1960), 254-264.

Authors' summary: "The quantum mechanical joint position-momentum quasi-distribution function is applied to the nuclear shell model. By introducing approximate quasi-position and quasi-momentum variables, the quasi-distribution function is converted into a non-negative (and hence nonquasi-) distribution. Numerical results are presented for one-dimensional and three-dimensional potentials leading in three dimensions to a nonisotropic nonindependent distribution with a predominance of low momenta at the nuclear surface. These results are in contrast with the usual Thomas-Fermi model and in addition provide a simple base for the discussion of direct nuclear reactions involving an average over many states of a residual nucleus for which linear momentum as opposed to angular momentum is a relevant quantity."

6522:

Lee, Kiuck; Inglis, D. R. Nuclear deformation in the spheroidal shell model. Phys. Rev. (2) **120** (1960), 1298-1302.

Authors' summary: "In the usual shell-model procedure, the effective Hamiltonian contains only half the sum of the shell-model potentials of nucleons in order to avoid counting average pairwise interactions twice. Because of the factor one-half, the nondiagonal elements of this Hamiltonian in the harmonic oscillator representation do not vanish, but they have been neglected in previous calculations of nuclear deformations by Nilsson and others, in which one minimizes total shell-model energy at constant volume. It is here shown in typical cases (without taking spin-orbit coupling into account) that the equilibrium deformation is unaltered in second and third order and that fourth-order modification arising from the nondiagonal elements is very small. The relation of these nondiagonal elements to those of the pairwise interactions is also discussed."

6523:

Fonda, Luciano; Newton, Roger G. Threshold effects in three-body channels. Phys. Rev. (2) **119** (1960), 1394-1399.

Authors' summary: "The possibility of obtaining threshold anomalies in reactions leading to three-particle channels is studied in detail. It is found that a threshold cusp or rounded step exists in reactions whose final three-body channels have at least one particle in common. The effect appears as a function of the momentum of the common particle while the total energy is fixed."

6524:

Baldin, A. M. Polarizability of nucleons. Nuclear Phys. **18** (1960), 310-317.

Author's summary: "Estimates of dipole polarizabilities of nucleons and the values they involve are given on the basis of data on photo-production of  $\pi$ -mesons and the Compton effect on nucleons. It is indicated that no upper estimate of neutron polarizability exists at present. The preliminary experimental data now available may be interpreted as indicating that a neutron has an abnormally large polarizability. The effects leading to the inapplicability of the impulse approximation for describing the reaction  $\gamma + d \rightarrow p + n + \gamma'$  are estimated. It is pointed out that the measurement of the cross section of the reaction  $\gamma + d \rightarrow d + \gamma'$  would yield an answer for the value of neutron dipole polarizability."

6525:

Berthelot, André. Les développements récents relatifs à la symétrie droite-gauche en physique. Nucleus **1960**, 356-367.

6526:

Fujii, Kanji. Symmetries of the interactions among strongly interacting particles and non-leptonic decays of hyperons. Progr. Theoret. Phys. **24** (1960), 259-270.

From the author's summary: "A symmetry law which can hold true in both strong and weak interactions among the strongly interacting particles is investigated and an example is given. The requirement of invariance under the proposed symmetry transformation and some assumptions lead to interactions which give results consistent with experiments on  $\Sigma^{\pm}$  and  $\Lambda$  decays."

P. W. Higgs (Edinburgh)

6527:

Kerimov, B. K. Nuclear saturation and the Lévy-Klein potential of pseudoscalar meson theory. Ž. Èksper. Teoret. Fiz. **32** (1957), 377-379 (Russian); translated as Soviet Physics. JETP **5**, 326-328.

6528:

Kurgelaidze, D. F. On the nonlinear theory of elementary particles. Ž. Èksper. Teoret. Fiz. **38** (1960), 462-474 (Russian. English summary); translated as Soviet Physics. JETP **11**, 339-346.

Author's summary: "The energies and momenta of spinor fields in theories with pseudovector and scalar nonlinear terms are calculated on the basis of a number of

new exact solutions of the wave type. By a semiclassical quantization the mass of the nucleon is determined as  $k_0 l = 2^{1/2} \pi^{3/2} \approx 7.84$ . The dependence of the energy of the field on the degree of nonlinearity is established. The method of fusion is used to derive from the nonlinear spinor equation a nonlinear undor equation, which on certain assumptions reduces to a nonlinear meson equation of the Klein-Gordon type. The conformal invariance of the nonlinear equations of the meson and spinor fields is discussed."

G. Feinberg (New York)

6529:

Salam, Abdus. Invariance properties in elementary particle physics. Lectures in theoretical physics (Boulder, Colo., 1959), pp. 1-30. Interscience, New York, 1960.

Contains a rather thorough and somewhat unconventional review of the representations of the 3- and 4-dimensional homogeneous and inhomogeneous Euclidean rotation groups and the homogeneous and inhomogeneous Lorentz groups. Applications to the systematization of strongly interacting particles are briefly mentioned.

P. Roman (Boston, Mass.)

6530:

Sakurai, J. J. Symmetry laws and elementary particle interactions. Lectures in theoretical physics (Boulder, Colo., 1959), pp. 31-209. Interscience, New York, 1960.

This is a very detailed survey of a host of problems related to the theory of elementary particle interactions. It discusses the various continuous space-time transformations that are linked to basic conservation laws, then illuminates and applies to many problems the concepts of parity, time reversal and charge conjugation transformations. The remarkable CPT-theorem is discussed from several angles. Special attention is given to the  $\gamma_5$  invariance of weak interactions. Various gauge transformations, such as the electromagnetic, baryon, and lepton conservation gauge transformations are discussed, and then the three-dimensional isospin formalism is introduced and applied to many problems. Special attention is paid to strangeness, and the selection rules pertaining to weak interactions.

The above listing gives only a vague notion of the broadness of field that is covered by this article. It can be highly recommended to students of the theory of elementary particles, provided they have already acquired a preliminary knowledge in this field. The ideas and methods are well illuminated and due emphasis is given to the proper points. The only drawback of the article is that it carries on it the mark of rough lecture notes and is, in places, somewhat unsystematic.

P. Roman (Boston, Mass.)

6531:

Fujii, Kanji; Itô, Daisuke. On multipole model of baryon-pion interactions. Progr. Theoret. Phys. 23 (1960), 815-820.

A model of baryon-pion interactions [D. Itô, S. Minami and H. Tanaka, Progr. Theoret. Phys. 22 (1959), 159-167; MR 22 #2393] in which the various baryons are described by a single non-local field is developed. In a multipole expansion of the non-local interaction the usual strong interactions appear as the monopole term, the weak interactions as the dipole term of the series.

P. W. Higgs (Edinburgh)

6532:

Gunson, J.; Taylor, J. G. Unstable particles in a general field theory. Phys. Rev. (2) 119 (1960), 1121-1125.

Authors' summary: "The problem of unstable particles in quantum field theory is treated as one of the interpretation of complex singularities appearing in the analytic continuation of scattering amplitudes into unphysical sheets of their Lorentz invariant variables. Suitable continuations are shown to hold under certain restrictive assumptions in a general field theory, making use of unitarity and causality of the  $S$  matrix. The extra singularities appearing in the continuation are fixed isolated poles, in accordance with a conjecture of Peierls."

G. Feinberg (New York)

6533:

Prigogine, I.; Leaf, Boris. On the field-matter interaction in classical electrodynamics. I. Physica 25 (1959), 1067-1079.

The authors consider the classical problem of the interaction of a point particle with the electromagnetic field. By using the statistical mechanical formulation for the many degrees of freedom of the field, and by expanding in powers of  $e^2$ , the self-accelerated solutions are avoided, and the usual (infinite) effective mass is obtained. A most interesting result is found by doing the same calculation, not in vacuum, but in a black-body radiation field at a finite temperature. Then the field-particle interaction is modified, and a temperature-dependent contribution to the mass appears. This serves in principle, but not in practice, to distinguish the electromagnetic from the other contributions to the particle's mass. This point, in principle, may be expected to carry over to quantum mechanics.

H. W. Lewis (Madison, Wis.)

6534:

Nakamura, Hiroshi. On a model on nucleon structure and meson-nucleon interaction. I. Progr. Theoret. Phys. 24 (1960), 271-290.

Author's summary: "The problems of  $\pi$ -nucleon collision and nucleon-antinucleon annihilation are treated in this paper (I) using a phenomenological model in which we assume the existence of the repulsive core for nucleon-nucleon potential and the attractive core for nucleon-antinucleon potential.

"The results give rather good agreement with experimental evidence not only with respect to  $\pi$ -nucleon collision but also to nucleon-antinucleon annihilation.

"We shall treat the problems of nucleon structure by using the same model in the following paper (II)."

6535:

Iso, Chikashi; Kawaguchi, Masaaki; Miyamoto, Yoneji. Composite model and the nature of vector interactions in strangeness-violating decays. Progr. Theoret. Phys. 24 (1960), 97-110.

6536:

Daiyasu, Kazuaki; Sugano, Reiji.  $K^+-K^0$  mass difference. Progr. Theoret. Phys. 23 (1960), 846-852.

Authors' summary: "In order to explain the  $K^0-K^+$

mass difference by means of electromagnetic interaction, the form factors in the Pauli term and in the electromagnetic-polarizability-term for meson are investigated. The magnitudes of contributions from these terms are estimated for the case of the exponential form factor and compared with that from nucleons."

6537:

Maki, Ziro. A note on the leptonic decay of hyperons. *Progr. Theoret. Phys.* **23** (1960), 853-858.

Authors' summary: "The leptonic decay modes of pions and  $K$ -mesons are studied from the viewpoint of compound model (Sakata model). By comparing these processes ( $\pi \rightarrow \mu + \nu$  and  $K \rightarrow \mu + \nu$ ), a conjecture for the leptonic decay of hyperons (e.g.,  $\Lambda \rightarrow p + \mu^- + \bar{\nu}$ ) is given which suggests that the (squared) bare coupling constant of this process is smaller than that of ordinary  $\beta$ -decay or  $\mu$ -capture process of nucleons by a factor  $\sim 10$ ."

6538:

Rockmore, Ronald M. Effects of particle-particle interaction on the moment of inertia of many-fermion systems. *Phys. Rev.* (2) **116** (1959), 469-474.

This paper gives the proof of a conjecture of Amado and Brueckner [*Phys. Rev.* (2) **115** (1959), 778-784; MR **21** #6252] that a large rotating system of interacting Fermions (subject to periodic boundary conditions) has the rigid-body moment of inertia. Amado and Brueckner showed this to be correct to first order in the interparticle interaction. It is shown here, that in the random phase approximation, the proof can be extended to all orders in the interaction. Use is made of the fact that the significant contributions to the moment of inertia arise from states close to the Fermi-surface. An equivalent Hamiltonian in terms of Boson operators [K. Sawada, *ibid.* (2) **106** (1957), 372-383; MR **19**, 98] is introduced. A single unitary transformation then suffices to display the rotational energy and to give the value of the moment  $T$ .

F. Villars (Cambridge, Mass.)

6539:

Iwamoto, Fumiaki; Yamada, Masami. Cluster development method in the quantum mechanics of many particle system. II. Saturation of nuclear forces. *Progr. Theoret. Phys.* **18** (1957), 345-356.

Authors' summary: "The cluster development method formulated in the first paper [*Progr. Theoret. Phys.* **17** (1957), 543-555; MR **19**, 79] is applied to the discussion on the saturation of nuclear forces. We assume Serber type two-body forces with hard cores, in which central potentials are assumed to be spin independent. The potential depth and range are determined from the scattering length and the effective range in the spin singlet state, and the hard core radius is taken as  $D = 0.6 \times 10^{-13}$  cm. The Coulomb forces are neglected. The usual variational method is used with a simple trial correlation function. By using the one- and two-nucleon clusters only, the energy minimum per nucleon is found to be  $-5.2$  Mev at  $r_0 = 1.1 \times 10^{-13}$  cm. At this minimum point the three-nucleon cluster terms are evaluated. They are one order of magnitude smaller than the leading term, and the convergence of the development seems to be fairly good. It is seen that the effective potential for

the single particle motion in the nuclear matter is much weakened after separating the short range correlations between nucleons."

6540:

Eleonskii, V. M.; Zyryanov, P. S. Contribution to the theory of collective motion of particles in quantum mechanical systems. *Ž. Eksper. Teoret. Fiz.* **32** (1957), 515-519 (Russian. English summary); translated as *Soviet Physics. JETP* **5**, 432-435.

Authors' summary: "A method for separating the collective motions in a system of interacting particles is presented. The connection between this method and earlier work by Zuburev, Bohm and Pines, and Tomonaga is established. An attempt is made to give a basis to the generalized heavy nucleus model."

## RELATIVITY

See also 6384, 6386, 6436, 6437, 6458, 6583.

6541:

Eddington, A. S. ★The mathematical theory of relativity. Cambridge University Press, New York, 1960. ix + 270 pp. \$2.95.

Paperbound reprinting of the 2nd [1924] edition.

6542:

Hillion, Pierre; Vigier, Jean-Pierre. Quantification du mouvement interne des masses fluides relativistes. *Cahiers de Phys.* **14** (1960), 109-117.

The authors review their own work as well as that of others on the internal motions of relativistic fluid masses. They discuss the classical theory of such motions as well as the quantizations of this theory. The quantum states associated with this theory are characterized by six quantum numbers. Three of these are used to characterize a family of particles and three to label each particle within a family. This classification is compared to that of Nishijima and Gell-Mann; similarities and differences between the classifications are pointed out.

A. H. Taub (Urbana, Ill.)

6543:

Halbwachs, F. General Lagrangian and canonical formalism describing the relativistic motion of a free rotating particle. *Acta Phys. Polon.* **19** (1960), 93-114.

This paper gives a discussion of a classical relativistic rotating particle in Lagrangian form. The actual Lagrangian discussed involves the four velocity vector of the center of matter of the particle and a set of three other mutually orthogonal four-vectors which are each orthogonal to the four velocity vector. These three vectors describe the "internal" motions of the particle. A discussion of a Hamiltonian formalism for this problem is also included in the paper. The work here reported differs from that done previously by the author and others on the theory of classical particles with spin in that the details of the "internal" motion are not considered.

A. H. Taub (Urbana, Ill.)

6544:

Loedel P., Enrique. The clock paradox and change of system. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. Serie Segunda. Rev. 6, no. 3, 23-54 (1959). (Spanish. English summary)

6545:

Dolginov, A. Z.; Topygin, I. N. Relativistic spherical functions. II. *Ž. Eksper. Teoret. Fiz.* 37 (1959), 1441-1451 (Russian. English summary); translated as Soviet Physics. JETP 10 (1960), 1022-1028.

This work deals with infinite-dimensional representations of the Lorentz group and is a continuation of an earlier work [A. Z. Dolginov, same *Ž.* 30 (1956), 746-755, supplement to 30, no. 4, 6; MR 18, 176] on finite-dimensional representations. The basis functions of the unitary representations are given, and matrices for the four-dimensional rotation operator are found. The Clebsch-Gordan expansion for the infinite-dimensional representations is obtained. These representations are applied to the study of a reaction of spinless particles of the form  $a+b \rightarrow c+d$ .  
N. Rosen (Haifa)

6546:

Dolginov, A. Z.; Moskalev, A. N. Relativistic spherical functions. III. *Ž. Eksper. Teoret. Fiz.* 37 (1959), 1697-1707 (Russian. English summary); translated as Soviet Physics. JETP 10 (1960), 1202-1208.

Two methods of constructing the irreducible representations of the Lorentz group are considered, that of Dolginov and Topygin [cf. preceding review] and that of I. M. Gel'fand and M. A. Naimark [Izv. Akad. Nauk SSSR. Ser. Mat. 11 (1947), 411-504; MR 9, 495]. The connection between them is found and some properties of the representations are investigated. An expansion of the wave function of a particle in terms of these representations is obtained.  
N. Rosen (Haifa)

6547:

Selepin, L. A. Contribution to the theory of relativistically invariant equations. *Ž. Eksper. Teoret. Fiz.* 37 (1959), 1626-1638 (Russian. English summary); translated as Soviet Physics. JETP 10 (1960), 1153-1161.

Suppose that  $\alpha_\mu \partial_\mu \psi + \kappa \psi = 0$  is a relativistically invariant wave equation; here  $\psi$  transforms according to a finite-dimensional representation of the Lorentz group. Equations of the above kind satisfying certain additional requirements motivated by the applications were discussed in great detail, for instance, by I. M. Gel'fand and A. M. Yaglom [same *Ž.* 18 (1948), 703-733; MR 10, 583]. The purpose of the present paper is to propose a new method to study the group-theoretical properties of these equations, and to simplify the deduction of the commutation relations satisfied by the matrices  $\alpha_\mu$ ; this is achieved by a discussion of the algebra generated by them. Certain conditions for irreducibility are given and examples are worked out.  
L. Pukanszky (College Park, Md.)

6548:

Popov, V. S. Rotation of the spin of a relativistic particle with a magnetic moment moving in an external

field. *Ž. Eksper. Teoret. Fiz.* 38 (1960), 1584-1588 (Russian. English summary); translated as Soviet Physics. JETP 11, 1141-1143.

Author's summary: "The problem of the rotation of the spin is solved for a relativistic particle that has a magnetic dipole moment and moves in an external electromagnetic field. The angular velocity of the rotation of the spin is determined in the rest system of the particle, which is rigidly fixed to its trajectory (analog of the Frenet axis system for a four-dimensional curve). The results obtained are significant for experiments made to measure the magnetic and electric moments of elementary particles."

6549:

Peres, Asher. Absence of bound states in a gravitational field. *Phys. Rev. (2)* 120 (1960), 1044.

The Klein-Gordon and Dirac equations for a particle in the gravitational field of a point mass are investigated. It is shown that the geometrical properties of the Schwarzschild metric prevent the normalization of any bound-state solutions.  
D. W. Sciama (Ithaca, N.Y.)

6550:

Siokos, Theod. Chr. Length contraction and time dilation of the general theory of relativity. *Prakt. Akad. Athēnōn* 33 (1958), 58-69, (1959). (Greek. English summary)

6551:

Siokos, Theod. Chr. The clock paradox of the general theory of relativity. *Prakt. Akad. Athēnōn* 33 (1958), 212-223, (1959). (Greek summary)

6552:

Siokos, Th. Ch. The clock paradox. *Supplements. Prakt. Akad. Athēnōn* 34 (1959), 242-249. (Greek)

6553:

Rindler, W. Hyperbolic motion in curved space time. *Phys. Rev. (2)* 119 (1960), 2082-2089.

It is a moot point whether motion with "uniform acceleration" in general relativity is a matter of definition or of derivation. L. Marder [Proc. Cambridge Philos. Soc. 53 (1957), 194-198; MR 19, 103] has given two possible definitions, which are discussed in an appendix to the present paper. The motion of a rocket whose blast is so arranged that in its rest frame it always has the same acceleration, acting in a fixed direction relative to the compass of inertia, should be uniformly accelerated. Also, in principle, it is fully determinate. In the flat space-time of special relativity such a rocket performs hyperbolic motion. The author recognizes the problem of dynamically obtaining the corresponding equations in curved space-time from Einstein's field equations, but circumvents it by "geometrically" generalizing hyperbolic motion. The status of this generalization is still that of a definition, though with every likelihood of dynamical validity. The equations are then solved in detail for the particular case of de Sitter space time. It is found that in this space time

a particle moving radially with uniform acceleration ultimately moves with constant relative velocity through the substratum; that there is a critical first fundamental particle (galaxy) on its line of motion which it will never overtake; that, in turn, a light signal emitted at or after a certain critical time will not catch up with the accelerating particle; and that, if a particle with a given available acceleration  $\alpha$  passes beyond a certain proper distance (the  $\alpha$  horizon) it can no longer return to its place of origin. Possible applications to intergalactic rocketry are examined.

D. W. Sciama (Ithaca, N.Y.)

6554:

Lichnerowicz, André. Ondes et radiations électromagnétiques et gravitationnelles en relativité générale. Ann. Mat. Pura Appl. (4) 50 (1960), 1-95.

Detailed survey of what may be called Riemann tensor radiation theory up to about Spring 1959, exploiting the far-reaching analogies which exist between gravitational and electromagnetic radiation. Chapters: (I) Generally relativistic electromagnetic waves; (II) Gravitational wave-fronts (algebraic structure, propagation); (III) Gravitational radiation defined (rather geometrically). The Bel-Robinson tensor. Pure radiation; (IV) Geodesic deviation. Physical interpretation of the Riemann tensor; (V) 5-dimensional generalizations (Kaluza-Klein and Jordan-Thiry theories); (VI) Quantization (in effect, of linearized gravitational theory only). Limitations: Most of the propagation theory of radiation fields (apart from wave-fronts) is too recently developed to have been included. The various canonical viewpoints are not discussed. Only a few of the radiative solutions of the full equations are mentioned.

F. A. E. Pirani (London)

6555:

Kruskal, M. D. Maximal extension of Schwarzschild metric. Phys. Rev. (2) 119 (1960), 1743-1745.

There is presented a particularly simple transformation of the Schwarzschild metric into new coordinates, whereby the "spherical singularity" is removed and the maximal singularity-free extension is clearly exhibited [cf. C. Fronsdal, Phys. Rev. (2) 116 (1959), 778-781; MR 22 #1402].

D. W. Sciama (Ithaca, N.Y.)

6556:

Kompaneec, A. S. Propagation of a strong electromagnetic-gravitational wave in vacuo. Z. Eksper. Teoret. Fiz. 37 (1959), 1722-1726 (Russian. English summary); translated as Soviet Physics. JETP 10 (1960), 1218-1220.

By considering a solution of the electromagnetic and gravitational field equations having a particular form, the author finds that the field possesses a system of rectilinear and parallel characteristics, so that the velocity of propagation of any small disturbance is constant. He is of the opinion that solutions which are Euclidean at infinity will therefore always be of a nature having a Euclidean analog. He therefore expresses doubt concerning the "geons" of Wheeler and his idea of the relation between charge and the topological connectivity of space [J. A. Wheeler, Phys. Rev. (2) 97 (1955), 511-536; MR 16, 756].

N. Rosen (Haifa)

6557:

Fok, V. A. Comparison of different coordinate conditions in Einstein's gravitation theory. Z. Eksper. Teoret. Fiz. 38 (1960), 108-115 (Russian. English summary); translated as Soviet Physics. JETP 11, 80-84.

The author claims to have shown: "the fact that the Einstein-Infeld equations coincide with the equations of motion in harmonic coordinates is to be explained not by saying that these equations allegedly do not depend on the coordinate conditions, but simply by the fact that the Einstein-Infeld coordinate system does not differ from the harmonic one in the approximation under discussion".

L. Infeld (Warsaw)

6558:

Ehlers, Jürgen. Exterior solutions of Einstein's gravitational field equations admitting a two-dimensional abelian group of isometric correspondences. Colloque sur la théorie de la relativité 1959, 49-57. Centre Belge Rech. Math., 1960.

It is shown that the Weyl and Levi-Civita solutions of the Einstein gravitational field equations [H. Weyl, Ann. Physik 54 (1917), 117-145; 59 (1919), 185-188] can be characterized by the fact that they admit a two-dimensional abelian group of isometries having certain properties. Some related solutions are also discussed.

N. Rosen (Haifa)

6559:

Skripkin, V. A. Relativistic corrections in the problem of a point explosion in an ideal gravitating continuous medium. Astr. Zh. 37 (1960), 284-296 (Russian. English summary); translated as Soviet Astr. AJ 4, 267-278.

The author discusses an approximate solution of the Einstein field equations for the gravitational field of a perfect fluid in the spherically symmetric case. The caloric equation of state for the fluid is taken to be the classical one,  $\epsilon = p/(\gamma - 1)\rho$ , where  $\epsilon$  is the rest specific internal energy,  $p$  the pressure,  $\rho$  the rest density and  $\gamma$  the ratio of specific heats. It is assumed that  $\gamma = 4/3$ . Shocks are allowed to be present. The approximating procedure involves expansions in powers of  $1/c^2$  where  $c$  is the velocity of light in vacuum. The equations governing the zeroth, first and second approximations are derived and solved for a specific example. The zero order solution is taken to be that given by Sedov for a point explosion in an ideal medium with  $\gamma = 4/3$  and subject to the Newtonian laws of gravitation. It is shown that solutions exist corresponding to a gravitational wave propagating in the un-disturbed medium in front of the shock wave.

A. H. Taub (Urbana, Ill.)

6560:

Newman, E.; Goldberg, J. N. The measurement of distance. Colloque sur la théorie de la relativité 1959, 37-41. Centre Belge Rech. Math., 1960.

Abbreviated version of Phys. Rev. (2) 114 (1959), 1391-1395 [MR 22 #600].

F. A. E. Pirani (London)

6561:

Hess, S.; Tischer, M. Die Bewegungsgleichungen eines speziellen Zweikörperproblems. Ann. Physik (7) 6 (1960), 15-24.

Relativistic equations of motion are derived for a particle inside a rotating spherical shell. This is analogous

to the earth in the universe. Relativistic corrections are pointed out for centrifugal and Coriolis forces. It is shown that the centrifugal force has a component in the direction of the axis of rotation.

*E. Pinney (Berkeley, Calif.)*

6562:

**Witten, Louis.** Invariants of general relativity and the classification of spaces. *Phys. Rev. (2)* **113** (1959), 357-362.

In recent years the geometrical properties of the Riemann tensor of the (4-dimensional hyperbolic normal Riemannian) space-time of general relativity theory have attracted some attention in connection with the physical interpretation of the theory. Hitherto they have been discussed—exclusively, as far as the reviewer is aware—in terms of tensors. The present author develops the interesting and suggestive idea of carrying out the geometrical analysis in terms of spinors instead. He expresses the Riemann tensor in terms of two 4th rank spinors (one of which vanishes in empty space-time) and constructs out of the latter the 14 2nd order scalar invariants of the metric tensor. The author's conjecture that his classification is equivalent to Petrov's [*Kazan Gos. Univ. Uč. Zap.* **114** (1954), 55-69; *MR* **17**, 892] is mistaken. There are some errors of detail which have been corrected by Penrose [see following review] in the course of a similar but more extensive analysis.

*F. A. E. Pirani (London)*

6563:

**Penrose, Roger.** A spinor approach to general relativity. *Ann. Physics* **10** (1960), 171-201.

An elegant and detailed exposition, in terms of Infeld-van der Waerden spinors, of the mathematical apparatus of gravitation theory, with emphasis on the geometrical theory of the Riemann tensor. The vacuum Riemann tensor (or, more generally, the Weyl conformal curvature tensor) corresponds to a totally symmetric 4-index spinor. This spinor is the symmetrized product of 1-index spinors which define the 4 "gravitational principal null directions" in space-time. These same directions were discovered in a different way by Debever [*C. R. Acad. Sci. Paris* **249** (1959), 1324-1326; *MR* **21** #7060]. Their arrangement determines the Petrov type of the Riemann tensor [*A. Z. Petrov, Kazan Gos. Univ. Uč. Zap.* **114** (1954), 55-69; *MR* **17**, 892].

The paper shares some basic ideas with that by Witten [see preceding review]. Some errors of detail in Witten's paper are corrected.

*F. A. E. Pirani (London)*

6564:

**Arnowitt, R.; Deser, S.; Misner, C. W.** Note on positive-definiteness of the energy of the gravitational field. *Ann. Physics* **11** (1960), 116-121.

"The energy of the gravitational field is shown to be positive-definite for the case where initially (a) the spatial metric  $g_{ij}$  can be made isotropic and (b) the second fundamental form  $K_{ij} = -(-g^{00})^{-1/2}\Gamma^0_{ij}$  (which is essentially  $\partial g_{ij}/\partial t$ ) is arbitrary apart from the coordinate condition  $K = K_{ij}g^{ij} = 0$  defining the initial  $t = \text{const.}$  surface. Physically, this situation corresponds to no wave excitations initially in  $g_{ij}$ , but arbitrary ones in  $K_{ij}$ . More precisely, the two canonical coordinates of the field are

initially zero, while their conjugate momenta are arbitrary. The total energy of the system is also positive when sources with positive energies are coupled to the gravitational field. Some aspects of the general problem (where  $g_{ij}$  is also arbitrary initially) are discussed." (Authors' summary).

The authors' proof uses the canonical variables which they defined in two earlier papers [*Phys. Rev. (2)* **116** (1959), 1322-1330; **117** (1960), 1595-1602; *MR* **22** #4505a, b]. Another class of gravitational systems (that with axial and time-reversal symmetry) has been shown to have positive-definite energy by Brill [same *Ann.* **7** (1959), 466-483; *MR* **21** #7056].

*P. W. Higgs (Edinburgh)*

6565:

**Witten, Louis.** Static cylindrically symmetric solutions of the Einstein-Maxwell field equations. *Colloque sur la théorie de la relativité 1959*, 59-77. Centre Belge Rech. Math., 1960.

The solutions are obtained by solving the field equations for space free of matter in the form first given by Rainich [*Trans. Amer. Math. Soc.* **27** (1925), 106-136]. Under the given symmetry conditions, the equations reduce to  $(R_0^0)^2 = (R_1^1)^2 = (R_2^2)^2 = (R_3^3)^2$  and  $R_0^0 + R_1^1 + R_2^2 + R_3^3 = 0$ ,  $R_{\mu}^{\mu}$  being the Ricci tensor.

Three distinct cases emerge: (i) that of a wire carrying a steady current, and placed along the axis of symmetry ( $Oz$ ); (ii) that referring to electric and magnetic fields both parallel to  $Oz$ , in which is embedded a neutral wire along  $Oz$ ; (iii) that of an electric line-charge along  $Oz$ . In all three cases the mass of the singularity along  $Oz$  plays an important part in determining the electromagnetic field, and in cases (i) and (ii) the mass cannot be put equal to zero without abolishing the field altogether.

*W. B. Bonnor (Urbana, Ill.)*

6566:

**Witten, Louis.** Initial value problem of the Einstein-Maxwell field. *Phys. Rev. (2)* **120** (1960), 635-640.

Author's summary: "On an initial hypersurface,  $x^0 = 0$ , in the presence of gravitation and source-free electromagnetism one can specify the metric tensor,  $g_{\mu\nu}$ , and its partial derivatives,  $g_{\mu\nu,0}$ , as well as the electromagnetic tensor,  $f_{\mu\nu}$ . These quantities must be specified so that on the initial hypersurface two of Maxwell's equations are satisfied and so that the components  $R_{\mu}^{\mu}$  of the Ricci tensor are proportional to the components  $T_{\mu}^{\mu}$  of the electromagnetic energy-momentum tensor. It is sometimes possible to specify a different electromagnetic tensor on the initial hypersurface which together with the old metric and Ricci tensors will describe a properly set initial value problem such that the geometry in advance of the initial hypersurface is different for the different electromagnetic fields. Thus, the Ricci tensor on the initial hypersurface does not always uniquely describe the geometry off the hypersurface in the Einstein-Maxwell theory. The conditions when this nonuniqueness exists are explicitly derived. An initial value problem could be set by specifying  $g_{\mu\nu}$  and  $g_{\mu\nu,0}$  on the initial hypersurface and deriving an appropriate  $f_{\mu\nu}$ ; however,  $g_{\mu\nu}$  and  $g_{\mu\nu,0}$  cannot be arbitrarily specified but are subject to one rather complicated constraint condition on the hypersurface."

*A. Peres (Haifa)*

6567:

Droz-Vincent, Philippe. Covariance générale et lois de conservation. C. R. Acad. Sci. Paris 250 (1960), 3582-3584.

The well-known results due to E. Noether concerning the invariance properties of the action integrals of the general theory of relativity [Weyl, *Space-time-matter*, Methuen, London, 1922, p. 233] are applied to action principles with general Lagrangians. The author makes systematic use of the Lie derivative [Schouten, *Ricci-calculus*, 2nd ed., Springer, Berlin, 1954; MR 16, 521; p. 102] and thus obtains expressions for integrals of quantities which satisfy definite conservation requirements.

H. Rund (Durban)

6568:

Kalčin, Nikola St. On certain singular solutions of Maxwell-Einstein equations which might represent light quanta. Izv. Bŭlgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk. Ser. Fiz. 7 (1959), 199-218. (Bulgarian. Russian and English summaries)

The paper suggests that photons may represent singular solutions of the Maxwell-Einstein equations. A solution of the Maxwell equations in free space is identified as the vector potential of the photon. From this, expressions for the energy of the photon and the spin of the photon are derived.

D. E. Spencer (Storrs, Conn.)

## ASTRONOMY

See also 6365.

6569:

Newcomb, Simon. ★A compendium of spherical astronomy: with its applications to the determination and reduction of positions of the fixed stars. Dover publications, Inc., New York, 1960. xviii + 444 pp. \$2.25.

Unaltered republication of 1st edition [Macmillan, New York, 1906]. A practical handbook, including a preliminary section on interpolation and theory of errors, a list of star catalogues, and various tables, as well as the general theory of computational astronomy.

6570:

Rösch, J. Des limites de l'observation astronomique, schéma théorique. Nucleus 1960, 401-404.

6571:

Garofalo, A. M. New set of variables for astronomical problems. Astr. J. 65 (1960), 117-121.

It is suggested that the commonly used elliptic elements of perturbation theory be replaced by 3 elements specifying the orbital plane and a reference direction therein, and  $h, \beta_1, \beta_2$ , where  $h$  is the angular momentum and  $\beta_1, \beta_2$  are rectangular components in the orbital plane of a vector in the direction of perihelion and of length  $ea/h^2$ , where  $e$  is the eccentricity and  $\mu$  is the attracting mass, in appropriate units. It is pointed out that the new differential equations are valid for all eccentricities and do not have some of the singularities of the familiar equations.

W. Kaplan (Ann Arbor, Mich.)

6572:

Message, P. J. On Mr King-Hele's theory of the effect of the Earth's oblateness on the orbit of a close satellite. Monthly Not. Roy. Astr. Soc. 121 (1960), 1-4.

In the theory of artificial satellite motion by D. G. King-Hele [Proc. Roy. Soc. London. Ser. A 247 (1958), 49-72; MR 20 #5683] the author defines a plane  $\alpha$  of constant inclination to the equator, rotating about the earth's axis in such a way that the satellite moves in it always in the same sense about the earth's center. This requirement cannot be satisfied unless the maximum northerly and southerly latitudes are the same, which they are not if oblateness perturbations having  $Je$  as a factor are taken into account. ( $2J/3$  is the coefficient of the second harmonic term in the earth's potential,  $e$  the orbital eccentricity.) The contradiction is indicated in King-Hele's theory by the fact that the expression for the derivative of the longitude of the ascending node contains singularities at the points of greatest northerly and southerly latitudes. Either one, but not both, of these singularities can be removed by a suitable choice of disposable constants. To remove the remaining singularity, it is necessary to forego the constancy of  $\alpha$ . The problem is solved by a new definition of  $\alpha$ , which now differs from a mean value  $\alpha_0$  by a periodic term, the coefficient of which is obtained. The only modification required to King-Hele's theory is in the periodic terms of order  $Je$  in the longitude of the ascending node  $\Omega$ . The mean value of  $d\Omega/dt$  is unchanged. Consequently, the coefficients of the harmonic terms in the earth's potential derived with the theory from observations of artificial satellites are not affected.

D. Brouwer (New Haven, Conn.)

6573:

Steins, K. [Šteins, K.]; Sture, S. [Stüre, S.]. On an example of the application of matrices to celestial mechanics. Latvijas Valsts Univ. Zinātn. Raksti 28 (1959), no. 4, 141-143. (Russian. Latvian summary)

Using matrix language the authors derive the equations of transformation of the Eulerian angles relative to the axis of symmetry of a spheroid to the angles relative to an instantaneous axis of rotation.

R. G. Langebartel (Urbana, Ill.)

6574:

Chandrasekhar, S. ★Principles of stellar dynamics. Enlarged ed. Dover Publications, Inc., New York, 1960. x + 313 pp. \$2.00.

This edition is an unabridged and unaltered republication of the work originally published in 1942 [Univ. Chicago Press, Chicago, Ill.] and reviewed in MR 4, 57. The following articles are also included and are unabridged and unaltered. "Dynamical Friction", Parts I, II and III, as originally published in 1943 by the Astrophys. J. in volumes 97, no. 2 and 98, no. 1, and reviewed in MR 4, 260 and 5, 18, "New Methods in Stellar Dynamics", as originally published in 1943 by the Ann. New York Acad. Sci., volume 45, article 3, and reviewed in MR 5, 191.

6575:

Garfinkel, Boris. On the motion of a satellite in the vicinity of the critical inclination. Astr. J. 65 (1960), 624-627.

The author treats the motion of an artificial satellite in the vicinity of a singularity of so-called critical inclination.

The method is based on the removal of the short-periodic terms from the Hamiltonian by von Zeipel's method. By expanding the energy integral into Taylor series, the author shows that the motions of two of Delaunay's variables,  $g$  and  $G$ , are identical with that of a simple pendulum. *Y. Kozai (Cambridge, Mass.)*

6576:

Grebenikov, E. A. Conference on problems in the mathematical theory of the motion of artificial celestial bodies. *Astr. Zh.* **37** (1960), 362-368 (Russian. English summary); translated as *Soviet Astr. AJ* **4**, 343-351.

6577:

Guier, W. H.; Weiffenbach, G. C. A satellite Doppler navigation system. *Proc. IRE* **48** (1960), 507-516.

The paper deals with the feasibility of Doppler navigation and its error analysis. The navigational error due to a constant frequency drift error can be minimized in using the whole of a Doppler curve: Frequency changes in the order of 1 to 10 parts in  $10^8$  per hour during observations still allow navigation to less than one mile accuracy. Errors in the navigation velocity have an effect of similar magnitude, provided ship navigation is contemplated. The main cause of error is seen in the ionospheric refraction of the satellite's radio signal. The refraction effects cannot be ignored for frequencies even as high as 500 mc/s. The authors devote a great part of their investigation to the influence of the ionosphere on satellite Doppler shift, thus touching a very interesting mathematical and physical problem. Their final conclusion arrives at an approximation of the observed Doppler shift by two terms neglecting errors of higher order. The first term is given by the time rate of change of the geometrical (straight line) slant range. The second term is considered the refraction contribution to the Doppler shift and put proportional to the time rate of change of the total number of electrons along the signal path. The authors show that such approximation is a very workable one for practical purposes. Their mathematical derivation is based on the assumption that the difference in length between the optical ray path and the straight line of the geometric path can be neglected. Since high frequencies are assumed, the paper does not consider the curvature of the ray path and the change of this curvature during the passage of the vehicle. A static ionosphere is assumed. The paper gives an approximation of the dependence of the vacuum Doppler shift on the geodetic position of a slowly moving ship navigator. *P. R. Arendt (Elberon, N.J.)*

6578:

Tischer, F. J. Propagation-Doppler effects in space communications. *Proc. IRE* **48** (1960), 570-574.

The Doppler shift  $\Delta\omega$  is expressed by the alterations of the phase angle  $\phi$

$$\Delta\omega = \nabla\phi\bar{v},$$

where  $\bar{v}$  is the velocity vector of the vehicle. Use of the

reciprocity theorem is made. The phase is expressed by the electrical field  $E(\bar{r})$  at the observer's position  $\bar{r}$  as

$$\phi(\bar{r}) = \text{Im} \left[ \int_0^s \frac{dE(\bar{r})}{ds'} / E(\bar{r}) ds' \right],$$

where  $ds'$  is the line element along the path of propagation. This results in

$$\Delta\omega = \nabla \left\{ \int_0^s \text{Im} \left[ \frac{dE(\bar{r})}{ds'} / E(\bar{r}) \right] ds' \right\} \cdot \bar{v}.$$

Giving  $s$  the direction of  $\bar{v}$ , assuming a frequency high enough so that the permittivity is close to 1, and writing the field as a series

$$E(\bar{r}) = E_0(\bar{r}) + \sum_{n=1}^{\infty} \Delta E_n(\bar{r}),$$

where  $E_0$  is the field distribution for empty space, one arrives finally at the approximation

$$\Delta\omega = \text{Im} \left[ \frac{d(E_0 + \Delta E_1)}{d(\bar{v}t)} / E_0 \right] \bar{v}.$$

Apparently this deduction neglects the influence of the alterations of the path length due to alterations in the medium. Two examples are given: for a vehicle travelling upwards in the direction of the wave propagation which is perpendicular to the assumed stratification, and for a vehicle travelling horizontally and perpendicular to an assumed vertical stratification. This latter case is called a transverse Doppler shift. The two cases are used to explain deviations from the free space Doppler shift which are to be expected in non-homogeneous media. Thus, the paper is a welcome explanation of the observed frequency scintillations of satellite signals.

*P. R. Arendt (Elberon, N.J.)*

6579:

Mestel, L. The magnetic and dynamical fields outside a proto-star. *Monthly Not. Roy. Astr. Soc.* **119** (1959), 223-248.

The problem of the formation of a star from a contracting gas cloud is one of the major problems of modern stellar cosmogony.

As Mestel and Spitzer pointed out in 1956 [same Not. **116** (1956), 503-514; MR **19**, 369] the presence of a magnetic field disturbs considerably the contraction of a gas cloud. The present paper is a detailed investigation of the behaviour of a contracting gas cloud in the presence of a magnetic field in case of no rotation.

The paper is divided in two parts. Part I describes the structure of equilibrium configuration; part II is an analysis of the dynamic problem.

In Part I, Mestel shows that the field equations and the boundary conditions in case of no rotation lead to three kinds of fields: (i) a poloidal field, with lines of force in meridian planes; (ii) a class of twisted fields; (iii) a class of partially force free fields. However, fields of class (i) or (ii) exhibit neutral points. In case (i) there is a neutral line in the equatorial plane; in case (ii) there are neutral points on the axis. In such regions, the magnetic field cannot resist the drag of inward flow and Mestel gives a rough description of the shape which the field takes.

The dynamic discussion in part II is based on the assumption that the plasma velocity  $v$  is parallel to the

field. Some solutions of the problem of accretion are given and compared to Bondi's solution [ibid. 112 (1952), 195-204; MR 14, 212]. Velocity is cut down to well below Bondi's velocity.  
E. Schatzman (Paris)

6580:

Kaplan, S. A.; Klimišin, I. A. Some notes on the emission of light by shock waves under cosmic conditions. *Astr. Zh.* 37 (1960), 281-283 (Russian. English summary); translated as *Soviet Astr. AJ* 4, 264-266.

Authors' summary: "A study is made of the heating of the gas before the front of a shock wave by the radiation emitted by the latter, using the theory of light scattering in a medium with a moving boundary. Formulas are given which may be used to calculate the temperature distribution in the heated zone, and the width of this zone. Equations describe the increase in the intensity of radiation before the wave reaches the surface, which is due to the penetration of radiation through the higher-lying layers."

6581:

Hoyle, F. A covariant formulation of the law of creation of matter. *Monthly Not. Roy. Astr. Soc.* 120 (1960), 256-262.

A covariant law is given for the creation field of matter. When the electromagnetic and nuclear fields, and the stress within matter, are omitted from the energy-momentum tensor, the Einstein equations possess an infinity of isotropic, homogeneous, stable, steady-state solutions. The possibility exists that, if the neglected terms were reinstated, the solutions would possess a slow secular drift.  
D. W. Sciama (Ithaca, N.Y.)

6582:

Naan, G. I. Über den gegenwärtigen Stand der kosmologischen Wissenschaft. *Acad. R. P. Romine. An. Romino-Soviet. Ser. Mat.-Fiz.* (3) 14 (1960), no. 1 (32), 154-194. (Romanian)

Translation of a Russian original [*Voprosy Kosmog.* 6 (1958), 277-329; MR 21 #1215].

6583:

Vaidya, P. C.; Shah, K. B. A relativistic model for a shell of flowing radiation in a homogeneous universe. *Progr. Theoret. Phys.* 24 (1960), 111-125.

The authors discuss spherically symmetric solutions of the field equations of general relativity under the conditions that: (i) for  $0 \leq r \leq R_1(t)$ , one has the Robertson cosmological line element; (ii) for  $R_1 \leq r \leq R_0$ , one has a nonvanishing radial energy flux given by  $T_1^4(g_{44}/g_{11})^{1/2} = T_1^4 - T_2^4$ , where the indices 1 and 4 correspond to the radial and time coordinates, respectively, and 2 and 3 to the angle coordinates; and (iii) for  $r \geq R_0(t)$ , one has again the Robertson line element which may however differ from that in (i) so far as the curvature and the time dependence are concerned.

Two particular solutions are presented and are interpreted to represent the history of an isotropic universe consequent to a complete conversion of matter into radiation in a certain region. However, the Maxwell

equations are not shown to be satisfied in (ii) and there is no empty or pure radiation zone which one can identify as the locale of complete conversion.

A. Raychaudhuri (Calcutta)

6584:

Metzner, A. W. K.; Morrison, P. The flow of information in cosmological models. *Monthly Not. Roy. Astr. Soc.* 119 (1959), 657-664.

Five models of the universe are considered in which the relations for the apparent brightness,  $b$ , of a source of radiation and for the number of sources,  $N$ , are simple functions of the red-shift,  $z$ . The models are those in which the scale factor  $R$  is an exponential function of the time (de Sitter, steady-state) or a power of the time (Einstein-de Sitter, Page, Dirac, Milne). The results are introduced into the formula  $C = W \log(1+b/N)$  for the rate of reception of information, where  $W$  is the band-width and  $N$  is the noise power (a confusing notation since  $N$  also means the number of sources). Plots of  $C(1+z)$  are given and the  $\int_0^z N(1+z)C(1+z) dz$  is also calculated. The formula (9a) for the number of sources,  $N(1+z)$ , in the steady-state model differs from that used by the proponents of this theory in having  $(1+z)^4$  instead of  $(1+z)^3$  in the denominator [H. Bondi, *Cosmology*, Cambridge Univ. Press, New York, 1952; MR 14, 912; pp. 146-147]. The error arises through an incorrect definition of the element of proper volume of space. There are consequential errors in formulae (14), (15) and in the first line of Table I.  
G. C. McVittie (Urbana, Ill.)

6585:

Peres, Asher. Neutrino and cosmology. *Progr. Theoret. Phys.* 24 (1960), 149-154.

In the first part of the paper the author gives a refinement of the Einstein-Friedman model of an expanding universe by taking into account the proper motion of particles. The conclusion is reached that the mean velocity of the particles in the universe must decrease with increasing radius of the universe.

It is then stated (but not proved) that the neutrinos in the universe must carry a total energy which is comparable to or even larger than the rest energy of the nucleons. Most of them must have been produced in the initial stage of the exploding universe and according to the preceding analysis their mean energy today would be so small (a few kev) as to escape detection with the inverse  $\beta$ -process. But if their lifetime is large compared to the age of the universe they would contribute appreciably to the rate of expansion of the universe (the Hubble constant) and their gravitational effect could in principle be observed.

J. M. Jauch (Geneva)

6586:

Argence, E. Contribution à l'étude de l'optique géométrique de l'ionosphère. *J. Méc. Phys. Atmos.* (2) 2 (1960), 7-37. (English and Spanish summaries)

The first chapter of this paper summarizes the Maxwell theory for isotropic absorbing plasmas, including gas-kinetic considerations. The second chapter gives a survey of the conventional treatment of geometric-optical approximations in stratified non-absorbing media with the aid of the W.K.B. method. The advantage of the derivation based on an expansion with respect to  $\omega^{-1}$  is shown

when deriving the conditions for the applicability of the W.K.B. procedure. This chapter also discusses the linear refractive-index profile, and the identity of wave-packet trajectories with those derived from the normals to the wave fronts. The chapter ends with an interesting example of ray trajectories, derived with the aid of a Hamilton-Jacobi differential equation, in a non-stratified medium.

The third chapter discusses isotropic absorbing stratified media. The concept of the eiconal function is extended here to absorbing media by assuming the ordinary eiconal equation  $(\nabla\phi)^2 = (\omega/c)^2 \epsilon'(z)$  also for a complex refractive index given by the square root of  $\epsilon'(z) = P(z) - iQ(z)$ . The real part of the eiconal thus defined determines the ray trajectories, the imaginary part paths tangent to the directions of highest attenuation. The W.K.B. method can easily be extended to these cases of non-negligible absorption (though we believe that its analysis in section 4 could have been clearer), and results in differential equations for the ray trajectories which depend on both  $P$  and  $Q$ . In section 6 these trajectories are shown to be identical with those derived from the direction of the Poynting vector, at least for standard waves with a linearly polarized  $E$  field and an elliptically polarized  $H$  field. The important inequality (64) shows how the attenuation may be computed, for small absorption, by a simple integration along the ray trajectory. Moreover, previous considerations on ray trajectories in the case of small absorption are criticized rightly.

This article contains many useful explicit formulas for those interested in propagation theories. A confusing printing error occurs in formulas (35) and (36) on page 21 where a factor  $x$  has to be added to the exponent.

H. Bremmer (Eindhoven)

#### GEOPHYSICS

See also 6184, 6212, 6213, 6215, 6282, 6334, 6577, 6650, 6661.

6587:

Palm, Enok; Foldvik, Arne. Contribution to the theory of two-dimensional mountain waves. *Geofys. Publ. Norske Vid.-Akad. Oslo* 21, no. 6, 30 pp. (1960).

Standard perturbation techniques are used as in the considerable literature on this problem, but a new and better estimate is made of the amplitudes of waves at heights of 20-30 km. where mother-of-pearl clouds are often observed. The airstream profile at high altitude is estimated from the best observations available. As has long been suspected, the waves can be a cause of the clouds. The mathematical representation of the airstream profile (wind and stability) is new, and known results about the kinds of profile which give lee waves in the troposphere are given new expression. By means of it a one-layer model can be chosen to represent the whole stream in many cases. The development of rotors with increasing mountain height is illustrated by an example.

The paper is written discursively and provides a valuable assessment of the problem in addition to the new details worked out.

R. S. Scorer (London)

6588:

Trubnikov, B. N. Calculation of the vertical non-homogeneity in a stream of air passing over an elevation.

Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. 1959, no. 4, 79-84. (Russian)

6589:

Wen, S. C. Generalized wind wave spectra and their applications. *Sci. Sinica* 9 (1960), 377-402.

This paper is a most interesting attempt to combine the techniques of Sverdrup and Munk with those of Neumann in order to forecast the growth of a wind wave spectrum. After a critical review of the work that has gone before, various assumptions are made concerning the resistance coefficient, the energy transferred by tangential forces, the sheltering coefficient, the energy transferred by normal forces, the viscosity coefficient, and energy dissipation. These assumptions are all acceptable considering the uncertainties involved. The exponential term in the theoretical spectrum used in the Pierson, Neumann, and James wave forecasting technique is then assumed to be either time or fetch dependent and to have a certain functional form with undetermined parameters. Equations that require a balance for each spectral contribution are then derived, and, by means of these, the parameters in the exponent are found. The result turns out to be two new spectral forms. One depends on the wind speed, and the wind duration, and defines a spectrum as a function of frequency given these quantities. A second depends on the fetch and the wind speed, and describes the spectrum as a function of frequency given these quantities.

The results are then compared with various observations, not of a spectral nature, that have been made for a storm in the East China Sea and for other situations. The results check out as well as most wave forecasting theories.

There are, however, some inadequacies in the paper. The graphs of the theoretical spectra do not appear to be correctly drawn. A little additional analysis actually shows that the partially developed state for a given duration and wind speed has a spectrum that is exactly the same as the fully developed state for some lower wind speed. The same is also true for fetch and wind speed considerations. These results do not seem to be borne out by more recent theoretical results in which it appears that the spectra cannot form a nested family of curves.

W. J. Pierson, Jr. (New York)

6590:

Kasahara, Akira. The numerical prediction of hurricane movement with a two-level baroclinic model. *J. Meteorol.* 17 (1960), 357-370.

The abstract of this paper describes its contents fairly well. The abstract follows.

"A steering method of predicting hurricane movement is formulated based upon a two-level baroclinic model. The upper steering-flow field is constructed from the pressure weighted mean of the 200- and 500-mb steering-height fields, and the lower steering field is constructed from the pressure-weighted mean of the 700- and 1000-mb steering-height fields. Here, the steering flow of a hurricane is defined as the residual field after eliminating the vortex pattern from the total-flow field.

"The evolutions of the upper and lower steering flows are predicted simultaneously by solving the two-level steering-flow vorticity equations. Based upon those steering-flow forecasts, the movement of a hurricane is

predicted with the use of an equation which is derived from a solution of two vortex vorticity equations. A side condition is imposed that the upper and lower vortex patterns should move with the same velocity in the corresponding steering flows.

"Ten cases of predicting the movement of hurricane 'Betsy' (August, 1956) up to 48 hr are presented. A preliminary comparison of the forecasts with those obtained from the barotropic model is also made."

In addition, a few points of interest might be brought out. The prediction scheme consistently predicted the track of the hurricane to lie to the south and east of the observed track. It also tended, in the one hurricane studied, to forecast the recurvature before it actually occurred. The average 24-hour error vector was approximately 90 miles long and pointed toward 100 degrees with reference to true north. It would appear, however, that the method has merit. If the average error vector could be shown to have a consistent bias on a larger number of examples, it could be removed so as to provide a useful forecasting procedure. *W. J. Pierson, Jr. (New York)*

6591:

**Knopoff, Leon; Gilbert, Freeman.** Diffraction of elastic waves by the core of the Earth. *Bull. Seismol. Soc. America* 51 (1961), 35-49.

Authors' summary: "The problem of the diffraction of a seismic pulse by the core of the Earth is investigated theoretically. The result is compared to that of diffraction by a half plane. The differences are striking. Laboratory model experiments have been performed to verify the theoretical approximations in their regions of validity, and to complement the theory elsewhere. The curves, thus obtained, of the theoretical amplitude distribution in the shadow of the Earth's core agree very well with the observations of Gutenberg. It is therefore concluded that diffraction is a completely adequate explanation for the amplitude distribution in the shadow zone."

6592:

**Bullen, K. E.** Note on cusps in seismic travel-times. *Geophys. J.* 3 (1960), 354-359.

In the case of a seismic travel-time curve with two cusps and a section between them on which the time is a triple-valued function of the distance, it is often found that large amplitudes occur near the cusps. It has been usual to suppose that this increase in intensity is a necessary feature of such cusps. The author shows by examples that this is not true. Discontinuities in the layering of the earth may be such as to cause the distance as a function of the ray-parameter to have extrema at which it is not stationary.

*A. Blake (Framingham, Mass.)*

6593:

**Yoshiyama, Ryoichi.** Propagation of surface waves and internal friction. *Bull. Earthquake Res. Inst. Tokyo* 38 (1960), 361-368. (Japanese summary)

An attempt is made to explain the observed variation of the amplitudes of the 'Mantle waves' by invoking the possibility of a variation of the internal frictional losses with the wave periods, which however, were neglected by earlier workers (Ewing, Press and Sato). An empirical

form is assumed for the dependence of the amplitudes on the epicentral distance, and numerical values of the parameters are obtained using Sato's results. The author points out that it is not easy to obtain unique values for the parameters used even to satisfy the results associated with only two earthquakes, but suggests that future studies on observations with high accuracy may yield useful information about internal frictional loss.

*S. K. Chakrabarty (Howrah)*

6594:

**Asano, Shûzô.** Reflection and refraction of elastic waves at a corrugated boundary surface. I. The case of incidence of SH wave. *Bull. Earthquake Res. Inst. Tokyo* 38 (1960), 177-197. (Japanese summary)

Following a method suggested by R. Sato, an attempt is made to estimate the nature of the reflected and refracted waves associated with a plane SH wave incident on the surface of separation of two different media, which is assumed to be undulatory only in one direction. The wave equation is solved with the surface of separation represented by  $Z=\zeta$  and  $\zeta$  is represented by a Fourier's series. The SH displacement components are expressed as a superposition of an infinite number of harmonic waves whose wave normals are restricted by spectrum relations of the type  $\sin \beta_n - \sin \beta = np(\sigma_1 h_1^2)^{-1/2}$ .

Results are obtained neglecting terms containing powers of  $\zeta$  above the first, and numerical values derived for the particular case of normal incidence by taking  $\zeta = C \cos px$ ,  $\mu_2/\mu_1 = 2.1$ , and  $v_{s1}/v_{s2} = 3.5/4.6$ . The variations of the amplitudes thus obtained with the nature of corrugation of the surface of separation are represented graphically, and a number of conclusions arrived at.

The author has realized from his results that his method of calculation is not valid for  $L/L_{s1} < 0.5$ , where  $L$  is the wave length of corrugation and  $L_{s1}$  that of the incident wave. The spectrum relations of the type mentioned above put a lower limit to the value of  $L/L_{s1}$  and  $L/L_{s2}$  which can be used, and the calculations of  $B_0$ ,  $D_0$ ,  $B_1$  and  $D_1$  are valid only when  $L/L_{s1} > 1.3$ . Similarly  $B_2$ ,  $D_2$  are valid only when  $L/L_{s1} > 2.6$ . As the author has overlooked this restriction, several of his observations on the trend of the different curves are not precise, particularly those numbered (4), (6), (8), (9) and (11). Some of the anomalies noted by the author himself will also disappear when the above restriction on the domain of  $L/L_{s1}$  is maintained.

*S. K. Chakrabarty (Howrah)*

6595:

**Danes, Z. Frankenger.** On a successive approximation method for interpreting gravity anomalies. *Geophysics* 25 (1960), 1215-1228.

Author's summary: "A new method for quantitative interpretation of gravity anomalies is presented. The disturbing body is represented by a finite number of vertical prisms arranged on a pre-determined, regular grid. The horizontal dimensions of the individual prisms are small enough that they can be approximated by vertical-line mass elements at the axis of the prisms. Formulas for gravity due to one prism are derived and, for the case of Gulf Coast salt densities, plotted. Gravity due to the whole body is an algebraic sum of the contributions of all prisms at the appropriate depths and distances.

"This method makes possible the direct interpretation

by successive approximations by introducing proper geologic limitations. All the numerical work can be conveniently done on a high-speed digital computer. The method is especially suitable for features with predominant vertical dimension such as salt domes and igneous plugs. It gives at least the same, and possibly higher, degree of accuracy as the graphical dot chart methods and, carried out on a digital computer, should be about two thousand times faster."

6596:

Colombo, Serge. Champs magnétiques cosmiques et théorie hydromagnétique de l'effet dynamo. *Cahiers de Phys.* 14 (1960), 55-67.

An historical review citing geomagnetism, and magnetic fields in the sun, stars, and the galaxy. This is followed by a reduction of Maxwell's equations for a homogeneous electric conductor with a Lorentz induction term to matrix form.

A. Herzenberg (Manchester)

6597:

Yukutake, Takesi. Stability and non-steady state of self-exciting dynamos. I. *Bull. Earthquake Res. Inst. Tokyo* 38 (1960), 1-12. (Japanese summary)

6598:

Yukutake, Takesi. Stability and non-steady state of self-exciting dynamos. II. *Bull. Earthquake Res. Inst. Tokyo* 38 (1960), 437-449. (Japanese summary)

Author's summary: "The variation of the magnetic field and the angular velocity with time are studied for two conducting spheres rotating in an infinitely extended conductor. Induction equations are approximated by those expanded up to the second time-derivatives of the magnetic field.

"In some special cases, the magnetic field was found to change its polarity with diminishing amplitudes. The periods of the damping oscillation are about  $1.3 \times 10^4$  years when the spheres of 500 km in radius are supposed to be rotating about 1100 km apart from each other in the earth's core. If the work done by the external torque is assumed to be  $10^{-9}$  erg/sec.  $\text{cm}^3$ , the induced magnetic field and the surface velocity of the sphere become about 1 gauss and 0.7 cm/sec., respectively."

A. Herzenberg (Manchester)

#### OPERATIONS RESEARCH, ECONOMETRICS, GAMES

See also A5578, A5601, A5638.

6599:

Gale, David. ★The theory of linear economic models. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1960. xxi + 330 pp. \$9.50.

In this new text the author has brought together a surprising variety of linear mathematics with appropriate applications. Chapter 1 introduces some typical examples of linear programming problems, states some of the

principal theorems, and provides some economic interpretations of these theorems.

The mathematical development begins in Chapter 2 with a succinct self-contained unfolding of the basic facts of real linear algebra, including the main theorems on systems of linear inequalities. The results are presented both in algebraic and in geometric form. In the first part of Chapter 3 these results are applied to obtain the fundamental duality theorems for linear programming and to lay the theoretical foundation for the simplex method. In the final section of the chapter an application is made to allocation of resources in a competitive economy.

The theory and format of the simplex method are developed in Chapter 4. The degenerate case is given an elegant treatment. The emphasis is clearly on theory rather than on any development of computational facility; in particular, there is no attention given to developing a format suitable for use with computing machines. The numerical examples that are provided are very simple and the person who wishes to become a practitioner in the simplex method will have to look elsewhere to round out the excellent theoretical background provided here with practical techniques.

Chapter 5 treats integral linear programming, i.e., problems in which the optimal vectors are required to have integral coordinates. There are some problems such as the transportation problem and the personnel assignment problems in which it can be proved that the optimal solution in the ordinary sense automatically has integer coordinates. In geometric language this means that the vertices of the region of feasibility all have integral coordinates. This chapter introduces a number of classes of problems which have this property, and the underlying mathematical theorems are included.

Chapters 6 and 7 treat two-person zero-sum games. There are many interesting examples worked out. The theoretical equivalence with linear programming is established; an approximation method (called here the "learning" method and also known as the method of "fictitious play") is explained and its convergence proved. Much of the now classical theory of solutions is included in these chapters.

Chapters 8 and 9 consider other linear economic models including such topics as balance of trade, flow of income, price equilibrium, Leontief models of production, the general linear production model, and efficient points. In these examples the linearity restriction limits the range of applicability, but the models treated have received considerable attention in recent years and an understanding of them is a useful foundation upon which to base a study of the more general non-linear models.

An important feature of the book is the substantial collection of problems at the end of each chapter. These are not mere routine exercises but contain substantial theorems extending the development in the body of the text. Since the text itself is very lean and meaty these problems should aid the reader to add some desirable intuitive "fat". The mathematical heart of the book is Chapter 2 and here the development is strictly limited to precisely those facts which will be needed later. The book would be more useful as a text if this chapter had been less clever mathematically and provided more in the way of intuitive discussion. The effort toward conciseness was overdone in the definition of convex cone (p. 52); the phrase "non-empty" should be inserted.

The typography and format are excellent and the mathematical community is indebted to the RAND Corporation for sponsoring this addition to its publication list.

R. M. Thrall (Ann Arbor, Mich.)

6600:

Nikaidō, Hukukane. Stability of equilibrium by the Brown-von Neumann differential equation. *Econometrica* 27 (1959), 654-671.

A proof of global asymptotic stability of equilibrium is given for an economic model which differs from some of those previously considered in that special care is taken that price variables never become negative. This is achieved by assuming a one-sided price adjustment mechanism, namely, that the rate of change of price of a good is proportional to the excess demand if this demand is positive, and zero otherwise. Thus absolute prices cannot decrease. Stability is then defined to mean that the set of normalized prices (obtained by dividing each absolute price by the sum of the prices) approaches some equilibrium price vector. The proof makes use of the Brown-von Neumann differential equation [*Contributions to the theory of games*, pp. 73-79, Princeton Univ. Press, Princeton, N.J.; MR 12, 514] but requires the following rather strong hypothesis: Letting  $\Delta p_i$  be the change in the price of the  $i$ th good and  $\Delta E_i(p)$  the corresponding change in the excess demand for this good, it is assumed that  $\sum_{i=1}^n \Delta p_i \Delta E_i(p) < 0$  for  $\Delta p$  sufficiently small. This assumption locally turns out to imply that it holds globally and also implies the uniqueness of equilibrium. It would therefore have been of interest to give some economic justification for the assumption, which is, for instance, much stronger than the often used weak axiom of revealed preference.

D. Gale (Providence, R.I.)

6601:

Morishima, Michio. A reconsideration of the Walras-Cassel-Leontief model of general equilibrium. *Mathematical methods in the social sciences*, 1959, pp. 63-76. Stanford Univ. Press, Stanford, Calif., 1960.

This paper adds to a very large number of recent articles on the stability of general equilibrium. As usual, the stability conditions depend on a somewhat arbitrarily chosen market adjustment behaviour.

T. Haavelmo (Oslo)

6602:

Hurwicz, Leonid. Optimality and informational efficiency in resource allocation processes. *Mathematical methods in the social sciences*, 1959, pp. 27-46. Stanford Univ. Press, Stanford, Calif., 1960.

The subject of this article is related to the important theoretical and practical problem of finding automatic, decentralized, decision schemes that possess certain optimal properties (such as "Pareto-optimality").

T. Haavelmo (Oslo)

6603:

Johansen, Leif. Rules of thumb for the expansion of industries in a process of economic growth. *Econometrica* 28 (1960), 258-271.

The author works with a general equilibrium model consisting of  $i=1, 2, \dots, n$  markets for consumption

goods  $X_i$ . There is a demand equation and a supply equation for each good. The factors of production are labour  $N_i$  and real capital  $K_i$ . The production function in each sector  $X_i = X_i(N_i, K_i)$  is Cobb-Douglas. Capital and labour can move freely between sectors. With a given total labour force  $N = \sum N_i$  and a given total stock of real capital  $K = \sum K_i$  the model determines the quantities produced of each consumption good  $X_i$ .

Consider then a dynamic expansion of  $N$  and/or  $K$ . The model is brought into a path of moving equilibrium. Larger quantities of each good  $X_i$  can be produced. The supply of  $X_i$  increases. At the same time consumer demand for  $X_i$  increases also, because national income  $Y$  of the model increases. The author is interested in the special case when these increases of supply and demand of  $X_i$  will exactly offset each other at the existing set of prices:

$$(1) \quad \frac{\partial X_i}{\partial Y} dY = dX_i(N_i, K_i) \quad (i = 1, 2, \dots, n).$$

The prices of  $X_i$  will then remain unchanged. Another way to state (1) is to say that the production of  $X_i$  will expand proportionately to the income elasticity  $E_i$  of demand for  $X_i$ :

$$(2) \quad \dot{X}_i/X_i = cE_i \quad (i = 1, 2, \dots, n),$$

where  $c$  is constant for all  $i$ . The author shows that this case will occur either when all sectors are equally capital intensive or when  $N$  and  $K$  expand at the same rate over time. But if neither of these two conditions is fulfilled, equation (1) will not hold. The coefficient  $c$  will vary between different sectors. Assuming that  $K$  expands faster than  $N$ , the author shows that  $c$  will be high for capital intensive industries and low for labour intensive industries. In an empirical example, taken from Norwegian industry, he finds  $c=0.024$  for services,  $c=0.039$  for dwellings.

S. O. Thore (Uppsala)

6604:

Morishima, Michio. Economic expansion and the interest rate in generalized von Neumann models. *Econometrica* 28 (1960), 352-363.

This paper contains a generalization of some known properties concerning the rate of expansion in models of balanced economic growth. In particular, the connection between the rate of interest and the rate of growth is studied. The generalization consists mainly in the use of inequalities instead of ordinary production functions, demand equations, etc.

T. Haavelmo (Oslo)

6605:

Koopmans, Tjalling C.; Bausch, Augustus F. Selected topics in economics involving mathematical reasoning. *SIAM Rev.* 1 (1959), 79-148.

This paper summarizes a course of lectures on aspects of economic theory selected on the basis of mathematical interest. The emphasis lies heavily on applications of the theory of convex sets, and in particular of convex polyhedra, to problems relating to efficient allocation of economic resources, as the following outline indicates: Existence and optimality of competitive equilibrium (14 pp.); equilibrium in international trade (2 pp.); equilibrium over time and capital theory (3 pp.); von

Neumann's linear growth model (6 pp.); linear activity analysis and some applications (15 pp.); macroeconomic dynamics (8 pp.); econometrics (10 pp.). The treatment of each topic includes a survey of the problem, sketches of one or more leading results, and a valuable list of references. Though the discussions in most cases are too concise to stand on their own, each is an instructive introduction to the literature of its subject, and the whole paper successfully conveys the flavor and methodology of recent research in mathematical economics. None of the results reported are novel, though some of the proofs are original.

R. Dorfman (Cambridge, Mass.)

6606:

Lange, Oskar. The output-investment ratio and input-output analysis. *Econometrica* 28 (1960), 310-324.

The author uses an open model with fixed production and investment coefficients and easily shows how the output-investment ratio  $\beta = \Delta X/I$ , in the obvious notation, is determined by the latter and the structure of investment. His discussion is carried out in value terms throughout, apparently at fixed prices, though we are not told how they are determined or are related to the shadow prices in his programming models.

In seeking to maximize  $\beta$  and through it the rate of growth of output subject to a minimum level of consumption of each good, the author implicitly assumes that it necessarily pays to invest all the resources at one's disposal, though he clearly realizes that this is not so. Output is defined gross, so that wheat, e.g., appears once as itself, again in flour, and again in bread. His attempt to apply the same analysis to income, defined net of such double-counting, fails because he implicitly assumes that the replacement ratios are constant, though he himself showed that they depended on the very structure of investment which he is trying to determine.

The author's apparent neglect of labour supply as a determinant of economic growth is rather surprising, especially in an input-output model. Yet his definitions of consumption and income, among other things, seems to imply that he does just this.

W. M. Gorman (Birmingham)

6607:

Cherubino, Salvatore. Sul concetto di Economia astratta. *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat.* (8) 26 (1959), 654-661.

The author considers a Leontief model of the economy. He first shows that for a static economy it is possible simultaneously to maximize labor cost and minimize consumption, the values being equal. He then shows that for a certain dynamic economy it is possible simultaneously to minimize the production component of the rate of change of prices and maximize the price component of the rate of change of production, the results being equal.

H. Rubin (E. Lansing, Mich.)

6608:

Salton, Gerard. A new method for the payment of bills and the transfer of credit. *J. Assoc. Comput. Mach.* 7 (1960), 140-149.

In order to facilitate registration of the payments of utility bills, a system is suggested where the utility

company in question is allowed to draw the amounts from the payer's bank account without previous notice.

P. Johansen (Copenhagen)

6609:

Stone, Richard; Rowe, D. A. The durability of consumers' durable goods. *Econometrica* 28 (1960), 407-416.

The quantity of a consumer durable purchased in any period  $q$  is the sum of replacement demand (= consumption - depreciation)  $u$  and net investment  $v$ . The Stone-Rowe model explains these two components separately. Consumption  $u$  is assumed to be a continuous process which uses up the existing stock  $s$  at a constant rate  $u = s/n + q/m$ , where  $n$  and  $m$  are constants. Net investment  $v$  is added to the opening stock  $s$  of the period in order to bring  $s$  to the desired level  $s^*$ . The level  $s^*$  is assumed to be determined by prices and income.

Using a method developed by Nerlove, the parameter  $n$  can be estimated within the model. The authors find  $n$  close to 2 for hardware and  $n$  slightly above 1 for furniture (annual data). These results would indicate that the economic life-length of these durables is not very much longer than one year!

The authors seem to fail to understand that this poor show must be taken as an indication that the model needs drastic improvement. For a good explanation of  $s^*$  a range of explanatory variables would probably be necessary, such as financial assets, age, income change, price expectations.

S. O. Thore (Uppsala)

6610:

Houthakker, H. S. Additive preferences. *Econometrica* 28 (1960), 244-257.

Let  $x_i$  stand for  $A$ 's consumption of the  $i$ th good,  $p_i$  for its price and  $\mu$  for his money income. Let  $z_i = \mu/p_i$ . The author considers the conditions under which his utility function can be written in the directly additive form  $\sum \phi_i(z_i)$  or the indirectly additive form  $\sum \psi_i(z_i)$ , and derives various useful elasticity formulae. He finds that the direct addilog form  $\sum \alpha_i x_i^{p_i}$  fits the British budget studies of 1953 quite well, at least for the large aggregates—food, clothing, rent, durables, and other—while the indirect addilog form  $\sum \alpha_i x_i^{p_i}$  is less successful.

W. M. Gorman (Birmingham)

6611:

Bouzit, Jean. Quelques aspects actuels de la recherche opérationnelle. *Les mathématiques de l'ingénieur*, pp. 13-30. *Mém. Publ. Soc. Sci. Arts Lett. Hainaut*, vol. hors série, 1958.

The author considers four modalities of operations research indicating their necessity. Operations research (1) works on high levels in decision making; (2) utilizes the team idea in conducting its research by bringing together individuals of varying technical backgrounds; (3) is interested in abstract forms of decisions insofar as they can be separated from a synthesis of diverse concrete experiences; and (4) calls on the fundamental sciences according to the pressure of practical needs, e.g., probability, abstract geometries, logic, the theory of economic calculus, etc.

Some techniques of operations research may be obtained by classifying the field according to decisions: (1) Those techniques of decisions adapted to complex situations are

treated by mathematical programming; (2) decisions adapted to randomly varying situations and for uncertainties are treated by stochastic processes; and, finally, (3) game theory treats decisions in the presence of several adversaries. *T. L. Saaty* (Silver Spring, Md.)

6612:

Tonge, Fred M. Summary of a heuristic line balancing procedure. *Management. Sci.* 7 (1960/61), 21-42.

Author's summary: "This paper presents a heuristic procedure for balancing production assembly lines and a computer program for carrying out that procedure. This research was undertaken to investigate the application of complex information processing techniques (as used in producing the Chess Machine and Logic Theorist) to a typical industrial problem.

"The assembly line balancing problem is stated as: Given an assembly process made up of elemental tasks, each with a time required per unit of product and an ordering with other tasks, what is the least number of work stations needed to attain a desired production rate?

"The heuristic procedure for assembly line balancing consists of three phases: (a) repeated simplification of the initial problem by grouping adjacent elemental tasks into compound tasks; (b) solution of the simpler problems thus created by assigning tasks to work stations at the least complex level possible, breaking up the compound tasks into their elements only when necessary for a solution; (c) smoothing the resulting balance by transferring tasks among work stations until the distribution of assigned time is as even as possible.

"The heuristics used in each phase are considered in some detail. Appropriate means for mechanizing such a procedure are discussed, and operating results of the program on actual problems are presented."

*D. Teichrow* (Stanford, Calif.)

6613:

Boulding, Kenneth E.; Spivey, W. Allen. ★*Linear programming and the theory of the firm*. With contributions by Sherrill Cleland, Hans. H. Jenny, Ching-wen Kwang, C. Michael White, Yuan-li Wu. The Macmillan Co., New York, 1960. ix + 227 pp. \$8.00.

A collection of seminar papers primarily for economists. Chapter 2 is an introduction to linear algebra, chapter 3 a standard presentation of linear programming theory, and chapter 4 a comparison mainly in graphical terms of output decisions and cost curves of the business firm in the short run as viewed from the classical and the linear programming standpoint.

*M. J. Beckmann* (Providence, R.I.)

6614:

Lefebvre, Louis. ★*Allocation in space: production, transport and industrial location*. Contributions to Economic Analysis, Vol. XIV. North-Holland Publishing Co., Amsterdam, 1958. xv + 151 pp. \$4.50; 17 guilders.

This short monograph is concerned with the principles of efficient allocation of economic activity and resources among different industries and different industrial locations, taking explicit account of the costs of transporting raw materials and final products. Previous work on allocation among industries recognized transportation costs implicitly if at all. Previous work on industrial

location treated this problem as if it were distinct from the problem of allocating resources among industries. In point of fact, these two allocation problems must be solved simultaneously if a fully efficient production plan is to be attained.

The analysis is presented in a sequence of models of increasing inclusiveness, of which the following is typical. Let  $\alpha = 0, 1, \dots, m$  be an index designating  $m$  final products (0 denotes transportation),  $\beta, \mu = 1, \dots, n$  be indexes designating  $n$  production sites,  $\gamma = 1, \dots, r$  an index designating  $r$  consumption sites (some or all of which may coincide with production sites),  $\delta = 1, \dots, s$  an index designating  $s$  transportable raw materials or factors of production. Let  $x_{\alpha\beta}$  denote the quantity of good  $\alpha$  produced at location  $\beta$  and delivered to consumption site  $\gamma$ ,  $V_{\alpha\mu}$  the quantity of productive factor  $\delta$  originating at location  $\mu$  and used at location  $\beta$  to produce good  $\alpha$ ,  $V_{\delta\mu}$  the supply of factor  $\delta$  available at site  $\mu$  without transportation. Braces will denote vectors, so that  $\{x_{\alpha\beta}\}$  is the vector of  $mnr$  components specifying the quantities of all goods produced at all production sites and delivered to all consumption sites.

The constraints on the allocation plan are as follows. (1) The quantity of each resource used at each site cannot exceed the supply available, or  $\sum_{\alpha} \sum_{\beta} V_{\alpha\mu} \leq V_{\delta\mu}$ , all  $\delta, \mu$ . (2) The total output of each good at each site cannot exceed the amount producible with the resources devoted to it, or  $\sum_{\gamma} x_{\alpha\gamma} \leq \varphi^{\alpha}(\sum_{\mu} V_{\alpha\mu})$ , all  $\alpha, \beta$ , where  $\varphi^{\alpha}(\cdot)$  is the production function for good  $\alpha$ . (3) The total demand for transportation, which is the function  $\tau(\{x_{\alpha\beta}\}, \{V_{\alpha\mu}\})$ , cannot exceed the amount that can be provided by the resources devoted to the transportation industry, or  $\tau(\{x_{\alpha\beta}\}, \{V_{\alpha\mu}\}) \leq \varphi^0(\sum_{\mu} V_{\alpha\mu})$ . ( $V_{\alpha\mu}^0$  lacks one superscript because the production of transportation services is not identified with locations.) (4) All variables must be non-negative.

Subject to these constraints, an efficient plan will maximize  $W = \sum_{\alpha} \sum_{\beta} \sum_{\gamma} w_{\alpha\gamma} x_{\alpha\beta}$ , where  $w_{\alpha\gamma}$  is the pre-assigned value of good  $\alpha$  at consumption site  $\gamma$ . Subject to suitable regularity and convexity assumptions, the maximization can be carried out by a variant of the Lagrange method applicable to inequality constraints. [H. W. Kuhn and A. W. Tucker, *Proc. 2nd Berkeley Symposium Math. Statist. Prob.*, 1950, pp. 481-492, Univ. Calif. Press, Berkeley, Calif., 1951; MR 13, 855.]

The Lagrangean form is

$$\begin{aligned} \Lambda = & \sum_{\alpha} \sum_{\beta} \sum_{\gamma} w_{\alpha\gamma} x_{\alpha\beta} \\ & - \sum_{\delta} \sum_{\mu} \lambda_{\delta\mu} (\sum_{\alpha} \sum_{\beta} V_{\alpha\mu} - V_{\delta\mu}) \\ & - \sum_{\alpha} \sum_{\beta} \nu_{\alpha\beta} (\sum_{\gamma} x_{\alpha\gamma} - \varphi^{\alpha}(\sum_{\mu} V_{\alpha\mu})) \\ & - \lambda^0 (\tau(\{x_{\alpha\beta}\}, \{V_{\alpha\mu}\}) - \varphi^0(\sum_{\mu} V_{\alpha\mu})). \end{aligned}$$

The  $\lambda, \nu$  appearing in this function are non-negative multipliers which can be interpreted economically as shadow or efficiency prices. Maximization of this form, which is straightforward, yields simultaneously optimal outputs of final products, optimal allocation of resources by industry and by location of use, and the system of prices corresponding to this production plan.

None of the substantive results obtained are novel. The contribution lies in expounding an engine of analysis whereby familiar doctrines of allocation among industries

and allocation among locations emerge together as coordinate consequences of an inclusive general equilibrium model.

R. Dorfman (Stanford, Calif.)

6615:

Dantzig, G. B.; Ramser, J. H. The truck dispatching problem. *Management Sci.* 6 (1959/60), 80-91.

The truck dispatching problem is that of finding an optimum routing of a fleet of delivery trucks between a bulk terminal and a large number of stations supplied by the terminal. It is a generalization of the traveling-salesman problem. The paper describes and illustrates a procedure, based on a linear programming formulation, for obtaining a near optimal solution.

M. M. Flood (Ann Arbor, Mich.)

6616:

Ghouila-Houri, Alain. Une généralisation de l'algorithme de Ford-Fulkerson relatif aux réseaux du transport. *C. R. Acad. Sci. Paris* 250 (1960), 457-459.

This note develops the maximum flow-minimum cut theorem of Ford and Fulkerson [*Canad. J. Math.* 8 (1956), 399-404; MR 18, 56] for a directed graph  $G$  constructed from a given finite directed graph  $G$  by splitting each point and each line of  $G$  into a countably infinite set of points and lines respectively.

F. Harary (Ann Arbor, Mich.)

6617:

Ghouila-Houri, Alain. Sur l'existence d'un flot ou d'une tension prenant ses valeurs dans un groupe abélien. *C. R. Acad. Sci. Paris* 250 (1960), 3931-3933.

Let  $\mathcal{A}$  be a family of subsets of an abelian group  $G$  satisfying the following conditions: (I<sub>1</sub>) If  $X \in \mathcal{A}$  then  $-X \in \mathcal{A}$ . (I<sub>2</sub>) If  $X, Y \in \mathcal{A}$ , then  $X + Y \in \mathcal{A}$ . (I<sub>3</sub>) If  $X \in \mathcal{A}$  and  $x \in X$ , then  $\{x\} \in \mathcal{A}$ . (I<sub>4</sub>) Every finite subfamily with non-empty pairwise intersections has a non-empty intersection. These properties are typically possessed by the intervals of an ordered group.

Let  $\Gamma$  be an oriented graph and  $c$  a 'capacity' function from edges  $U$  of  $\Gamma$  into  $\mathcal{A}$ . A 'flow' is a function  $\varphi$  from  $U$  to  $G$  such that at each vertex  $s$  the sum of the values of  $\varphi$  on edges from  $s$  equals the sum on edges into  $s$ . A function  $\theta$  from  $U$  to  $G$  is called a 'tension' if there is a function  $\pi$  on vertices such that  $\theta(u) = \pi(s') - \pi(s)$ , where  $s$  and  $s'$  are the initial and terminal edges of  $u$ . A flow or tension is called 'compatible' with  $c$  if  $\varphi(u)$  or  $\theta(u)$  is in  $c(u)$ .

For any subset  $A$  of vertices define  $U_{A^+}$  to be the edges from  $A$  to its complement and  $U_{A^-}$  the edges to  $A$  from its complement. For any cycle  $C$  define  $U_{+C}$  to be the edges of  $C$  oriented in the direction of  $C$  and let  $U_{-C}$  be those oppositely oriented.

Theorem 1: A capacited graph admits a compatible flow if and only if for every set  $A$  of vertices

$$0 \in \sum_{u \in U_{A^+}} c(u) - \sum_{u \in U_{A^-}} c(u).$$

Theorem 2: A capacited graph admits a compatible tension if and only if for every cycle  $C$

$$0 \in \sum_{u \in U_{+C}} c(u) - \sum_{u \in U_{-C}} c(u).$$

Special cases for particular groups  $G$  and families  $\mathcal{A}$  give theorems of Hoffman, Roy and König. Proofs are by induction, which is somewhat unusual for this kind of problem.

D. Gale (Providence, R.I.)

6618:

Dickson, J. C.; Frederick, F. P. A decision rule for improved efficiency in solving linear programming problems with the simplex algorithm. *Comm. ACM* 3 (1960), 509-512.

The Simplex Method requires that the variable  $x_j$  to be introduced into the basis should be one for which  $z_j - c_j < 0$  (for a maximization problem). This criterion allows choice and it is a common practice to choose an  $x_j$  for which  $z_j - c_j$  is smallest. This paper argues for choosing instead, among the  $x_j$  for which  $z_j - c_j < 0$ , one for which

$$r_j = \frac{(z_j - c_j)^2}{(z_j - c_j)^2 + \sum_i (a_{ij}^+)^2}, \quad a_{ij}^+ = 0 \text{ when } a_{ij} \leq 0, \\ a_{ij}^+ = a_{ij} \text{ when } a_{ij} > 0,$$

is a maximum (where the  $a_{ij}$  give the dependence of the  $j$ th column on the  $i$ th basis vector). A similar criterion is given for the Dual-Simplex Method. The rule is rationalized geometrically. It is asserted that in computing experience only as many iterations are required as the number of basic variables or non-basic variables, whichever is smaller. Meanwhile application of the criterion involves only a 10% increase in the computation time for each iteration.

J. J. Stone (Menlo Park, Calif.)

6619:

Hanson, M. A. Errors and stochastic variations in linear programming. *Austral. J. Statist.* 2 (1960), 41-46.

Let an optimal base be given in a linear programming problem and let the elements of the matrix of equalities ( $A = [a_{ij}]$ ) be subject to small independent errors, as well as the availabilities ( $b_i$ ). Then an approximation formula gives the mathematical expectation of the activities:  $E x_k = \bar{x}_k + \sum_i \sum_j A_{ik} A_{ij} \bar{x}_j \sigma^2(a_{ij})$ , where the  $\bar{x}_j$  are computed from the mean values and the  $A_{ij}$  are elements of the inverse  $A^{-1}$ . The variance is

$$\sigma^2(x_k) = \sum_i \sigma^2(b_i) A_{ik}^2 + \sum_i \sum_j \sigma^2(a_{ij}) A_{ik}^2 \bar{x}_j^2.$$

Making the assumption that the errors are normally distributed and neglecting overlapping probabilities, it is also possible to derive the probability of choosing a wrong base, by similar methods. A numerical example is given.

G. Tintner (Ames, Iowa)

6620:

Gomory, Ralph E.; Baumol, William J. Integer programming and pricing. *Econometrica* 28 (1960), 521-550.

The authors describe Gomory's Method of Integer Forms, first presented in *Bull. Amer. Math. Soc.* 64 (1958), 275-278 [MR 21 #1230]; and they show how to use this algorithm to derive shadow prices for scarce indivisible resources in an integer programming problem.

One set of prices is obtained from the coefficients of the nonbasic variables in the final expression for the objective function. This may involve giving zero prices to goods that would nevertheless be useful if available in larger quantities. The positive prices tend to be awarded to the "artificial goods" whose limited availability shows up in the new inequalities.

But, apart from the constant term, the new inequalities can be expressed as weighted sums of the original inequalities. This is used to derive an alternative set of

prices in which the prices of the artificial goods are imputed back to the original goods.

Both sets of prices are shown to have some, but not all, of the economically significant properties of shadow costs in linear programming. *E. M. L. Beale (Teddington)*

6621:

Lambert, F. Programmes linéaires mixtes. Cahiers Centre Études Rech. Opér. 2, 47-126 (1960).

This paper presents a concise survey in French of the present state of mixed linear programming studies. It starts with an outline of the general problems and methods of solution in common use. Then there follows a short outline of Gomory's method for the solution of problems in integer linear programming.

The next section deals with the subject of mixed linear programming, that is problems in which some variables are restricted to integral values only, whilst other variables are not so restricted. The paper concludes with notes about some of the difficulties which arise in the solution of such problems.

There are also sections on non-convex programming and some non-linear programming problems which arise in economics. The paper includes many worked examples, illustrations and references, and it will certainly be a valuable introduction to the subject for any reader of French. The following paper by E. Morlet on the coding problems involved should be read with it [see following review]. *L. J. Slater (Cambridge, England)*

6622:

Morlet, E. Codage de la résolution de programmes linéaires mixtes. Cahiers Centre Études Rech. Opér. 2, 127-160 (1960).

This paper, written in French, deals with the problem of coding for an electronic computer the mixed linear programming problem discussed in the paper by F. Lambert [see preceding review]. The actual computer used was the IBM 650, so that the coding of the calculation is expressed in terms of the auto-code called "Fortranait", which has been developed by the IBM company.

This paper contains a short description of Fortranait, and then goes on to discuss the details of the linear programming problem itself. There are several excellent and helpful flow diagrams of the calculation. However, the underlying logical problem is that of expressing the actual details of a complicated calculation in terms which can be understood by any scientist, even if he is not familiar with the actual code of the computer being used. An auto-code is an attempt to do just this, but as there are almost as many auto-codes as there are automatic calculators their usefulness is somewhat limited. *L. J. Slater (Cambridge, England)*

6623:

Land, A. H.; Doig, A. G. An automatic method of solving discrete programming problems. *Econometrica* 28 (1960), 497-520.

This paper describes a method of solving linear programming problems when some of the variables (denoted by the vector  $x$ ) must take integer values.

One first solves the problem ignoring these integral constraints. One then constructs a tree of linear pro-

gramming problems such that all branches emanating from a vertex impose the additional constraint that some particular  $x$ -variable takes consecutive integral values.

The authors define a systematic method of exploring enough of this tree to either find the best solution to the original problem, or else establish that no such solution is possible. *E. M. L. Beale (Teddington)*

6624:

Theil, H.; van de Panne, C. Quadratic programming as an extension of classical quadratic maximization. *Management. Sci.* 7 (1960/61), 1-20.

The authors describe a procedure for maximizing a strictly concave quadratic function of variables subject to linear inequalities. This is based on the fact that the solution is obtainable by imposing some of the constraints as equalities, and ignoring the others.

One first solves the problem ignoring the constraints. One then constructs a tree of problems such that all branches emanating from a vertex impose an additional equality constraint for which the corresponding inequality was not satisfied at the vertex. When a feasible solution to the original problem is obtained, it must be the optimal solution unless some single (equality) constraint can be removed without violating the corresponding inequality.

{This approach can be used for any concave differentiable objective function, but it is based on a premise that the reviewer does not accept: that equality constraints are computationally much easier to handle than inequalities.} *E. M. L. Beale (Teddington)*

6625:

Maekawa, Takeshi. On B. Klein's method for nonlinear programming. *Bull. Univ. Osaka Prefecture Ser. A* 7 (1959), 177-180.

The author is concerned with maximizing  $f(x_1, \dots, x_n)$  subject to the constraints  $f_i(x_1, \dots, x_n) \leq c_i$ ,  $x_j \geq 0$ .

Hancock [*Theory of maxima and minima*, Ginn, Boston, 1917; pp. 149-150] suggested replacing  $f_i \leq c_i$  by  $f_i = c_i + x_{n+i}^2$  and using classical Lagrange multiplier techniques. Klein [*Operations Res.* 3 (1955), 168-175; MR 16, 937] showed that this leads to sets of alternative conditions for the constrained maximum.

The author shows how the alternatives correspond to different possible positions of the maximum. He also uses these to give another proof of the famous Kuhn-Tucker conditions [Kuhn and Tucker, *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 1950, pp. 481-492, Univ. of California Press, Berkeley, Calif., 1951; MR 13, 855]. But it is not clear that he establishes the non-negativity of the Lagrange multipliers. *E. M. L. Beale (Teddington)*

6626:

Wegner, Peter. A non-linear extension of the simplex method. *Management. Sci.* 7 (1960/61), 43-55.

The author has so little control over his notation and terminology that one cannot be sure of his meaning. But the general idea seems to be as follows:

One is trying to maximize a linear function of  $m$  non-negative variables subject to  $n$  nonlinear equality constraints. One finds a basic feasible solution (b.f.s.), i.e.,

a trial solution where  $n$  "basic" variables are positive and  $m-n$  "nonbasic" variables are zero. One computes the partial derivative of the objective function with respect to each nonbasic variable—keeping the others constant and allowing the basic variables to vary so as to continue to satisfy the constraints. If any partial derivative is positive, one finds a neighbouring b.f.s. by making the corresponding nonbasic variable positive in place of one of the basic variables. This involves guessing which variable should be made nonbasic, and solving  $n$  nonlinear simultaneous equations iteratively by the Newton-Raphson method. This new b.f.s. may or may not be better than the old.

Eventually one obtains a b.f.s. that cannot be improved by going to any neighbouring b.f.s. If all the partial derivatives are negative, this is a local optimum. Otherwise one assumes that the objective function is an approximately quadratic function of the nonbasic variables between the best b.f.s. and its neighbours. The author assumes that the maximum of this function, conditional on putting all nonbasic variables with negative partial derivatives at the best b.f.s. equal to zero, corresponds to a feasible solution. He also seems to assume that no variable that is nonbasic at the best b.f.s. and has a negative partial derivative there can be positive at the optimal solution.

*E. M. L. Beale (Teddington)*

6627:

Berge, Claude. Sur l'équivalence du problème du transport généralisé et du problème des réseaux. *C. R. Acad. Sci. Paris* 251 (1960), 324-326.

The author is concerned with a directed graph including a source  $B'$  and a sink  $B$ , and functions  $f_i(x_i)$  representing the cost of sending  $x_i$  units along arc  $i$ , where  $b_i \leq x_i \leq c_i$ . He defines the generalized transportation problem, (1), as the minimization of  $\sum f_i(x_i)$  for a flow pattern giving the maximum flow from  $B'$  to  $B$  such that the net flow into each other vertex is zero. He defines the fundamental network problem, (2), as the above when the minimization is replaced by the existence of "potentials" for each vertex of the graph such that the derivative  $f'_i(x_i)$  of the cost on each arc equals the potential difference along it, with appropriate modifications if  $x_i = b_i$  or  $c_i$ . He proves that any solution of (1) solves (2). Conversely, if the functions  $f_i$  are convex, any solution of (2) solves (1).

The author's proof uses the Kuhn-Tucker conditions. [A proof on the following lines is also possible: One replaces the bounds on  $x_i$  by making  $f_i$  increase indefinitely outside the permitted range. Then, given a flow pattern that minimizes  $\sum f_i$  for some given total flow from  $B'$  to  $B$ , one starts from  $B'$  and assigns potentials to each vertex in turn, using the values of  $f'_i$  along arcs from previously considered vertices. If this gave different potentials for any vertex  $V$ , then  $\sum f_i$  could be reduced by increasing the flow from  $B'$  to  $V$  via the arc giving the lowest potential at  $V$ , at the expense of some other route.]

[On p. 325, line 5, "un arc  $B$ " should read "un arc  $BD$ ". Although  $f'_i$  denotes the derivative of  $f_i$ , elsewhere dashes are simply superfixes.]

*E. M. L. Beale (Teddington)*

6628:

Franklin, J. N. The range of a fleet of aircraft. *J. Soc. Indust. Appl. Math.* 8 (1960), 541-548.

Author's summary: "The problem discussed in this

paper is to determine the range of a fleet of  $n$  aircraft with fuel capacities  $g_i$  gallons and fuel efficiencies  $r_i$  gallons per mile ( $i=1, \dots, n$ ). It is assumed that the aircraft may share fuel in flight and that any of the aircraft may be abandoned at any stage. The range is defined to be the greatest distance which can be attained in this way. Initially the fleet is supposed to have  $g$  gallons of fuel.

"A theoretical solution is obtained by the method which Richard Bellman calls dynamic programming. Explicit solutions are obtained in the case of two aircraft with different fuel capacities and fuel efficiencies and in the case of any number of aircraft with identical fuel capacities and identical fuel efficiencies.

"The problem is similar to the so-called jeep problem. The jeep problem was solved rigorously by N. J. Fine. A solution was also obtained by O. Helmer. Fine cited an unpublished solution by L. Alaoglu. The problem was generalized by C. G. Phipps. Phipps informally developed the special result which is deduced in § 4 of this paper."

*N. J. Fine (Philadelphia, Pa.)*

6629:

Moore, C. J.; Lewis, T. S. Digital simulation of discrete flow systems. *Comm. ACM* 3 (1960), 659-660.

This paper briefly examines the simulation of discrete flow systems as illustrated in (1) package-handling (arrival, loading, storage and departure of trucks) to study costs, and (2) air traffic control to compare several alternative route and altitude structures over a range of traffic densities.

*T. L. Saaty (Silver Spring, Md.)*

6630:

Sarf, Herbert. The optimality of  $(S, s)$  policies in the dynamic inventory problem. *Mathematical methods in the social sciences*, 1959, pp. 196-202. Stanford Univ. Press, Stanford, Calif., 1960.

In a dynamic inventory problem we have ordering cost consisting of unit plus reorder cost. If holding and shortage costs are linear, the optimal policy in each period is: If inventory  $x < s$ , order  $(S-x)$ ; if  $x > s$ , order zero. The policy is optimal for zero lag and also for time lag in delivery. The proof uses the notion of  $K$ -convexity: Let  $K \geq 0$ ,  $f(x)$  differentiable. Then  $f(x)$  is  $K$ -convex if  $K + f(a+x) - f(x) - af'(x) \geq 0$  for all positive  $a$  and all  $x$ . If  $f(x)$  is not differentiable, then the condition is

$$K + f(a+x) - f(x) - a[f'(x) - f'(x-b)]/b \geq 0.$$

*G. Tintner (Ames, Iowa)*

6631:

Вильямс, Дж. Д. [Williams, J. D.]. ★Совершенный стратег: или букварь по теории стратегических игр. [The compleat strategist: being a primer on the theory of games and strategy]. Translated from the English by Yu. S. Golubev-Novozhilov; edited by I. A. Poletaev. Izdat. "Sovet. Radio", Moscow, 1960. 269 pp.

The original (McGraw-Hill, 1954) was listed in MR 15, 812.

6632:

Mills, Harlan. Equilibrium points in finite games. *J. Soc. Indust. Appl. Math.* 8 (1960), 397-402.

An algebraic characterization of equilibrium points in

finite games is given. From this characterization the author shows that: (a) equilibrium points can be described in terms of solutions to certain non-linear programming problems involving linear constraints and bilinear forms in the objective functions; (b) in the two-person case the equilibrium points can also be described as solutions of a system of linear inequalities where certain variables must be integral.

H. Raiffa (Cambridge, Mass.)

6633:

Goldman, A. J.; Stone, J. J. A continuous poker game. *Duke Math. J.* **27** (1960), 41-53.

The rules of the game are as follows: After each of the two players ante 1 unit and are given a single random number from  $[0, 1]$ , player 1 can fold or bet an amount  $b-1$ ; player 2 can then drop, see, or raise by betting  $a-1$  (where  $a > b$ ); player 1 can then fold or see; high number wins. The authors find the value of the game and characterizes the sets of optimal strategies and of semioptimal strategies (those which yield the value of the game against every optimal strategy of the opponent). A heuristic argument leading to the "guessing" process used in solving the game is given. H. Raiffa (Cambridge, Mass.)

6634:

Shubik, Martin; Thompson, Gerald L. Games of economic survival. *Naval Res. Logist. Quart.* **6** (1959), 111-123.

Two strategic games of economic interest are explored in detail. The one-person survival game concerns a corporation whose fortune at time  $t$ ,  $t=0, 1, 2, \dots$ , is  $x_t$ . For each value of  $t$  the corporation decides on a non-negative integral dividend payment,  $w_t$ . With the remainder of its fortune,  $x_t - w_t$ , it engages in a sub-game in which it has probability  $p$  of winning 1 unit and probability  $1-p$  of losing 1 unit. The game stops when the fortune falls to zero, and the value of the pay-off is the discounted sum of the dividends,  $\sum \rho^t w_t$ ,  $\rho < 1$ . What is the dividend payment at each stage of the game, i.e., for each value of  $x_t$ , that maximizes the expected value of the pay-off? The solution involves a functional equation of the type made familiar by Bellman's *Dynamic programming* [Princeton Univ. Press, Princeton, N.J., 1957; MR **19**, 820], which in this instance reduces to a solvable difference equation. More general variants of this game, and its economic interpretation, are discussed.

The second game is a two-person game of economic survival, in which two corporations with initial fortunes  $x$  and  $y$  play at each value of  $t$ ,  $t=1, 2, \dots$ , a matrix game with matrix

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The game continues until the fortune of one of the corporations is exhausted, at which time the other corporation obtains a prize whose value may depend on  $x$  and  $y$ . The optimal strategies are functions of  $x$  and  $y$  and, for some values, surprising. Economically this game is a schematization of a contest between a pair of duopolists, each trying to drive the other out of the market. Generalizations are discussed briefly.

R. Dorfman (Cambridge, Mass.)

## BIOLOGY AND SOCIOLOGY

See also 6098.

6635:

Mansfield, Edwin; Hensley, Carlton. The logistic process: tables of the stochastic epidemic curve and applications. *J. Roy. Statist. Soc. Ser. B* **22** (1960), 332-337.

A logistic process is defined as follows. A "diseased" individual is introduced into a population. The chance that any uninfected individual should catch the disease in a short time interval of length  $\Delta$  is  $B\Delta$  multiplied by the number of individuals already infected.  $B$  is a constant, conveniently standardized as 1 by changing the time scale.  $m(t)$  is the expected number of infected individuals at time  $t$ , and  $z(t) = dm(t)/dt$ . Tables are given of  $z(t)$  for population size 5 through 40.

C. A. B. Smith (London)

6636:

Watterson, G. A. Non-random mating, and its effect on the rate of approach to homozygosity. *Ann. Human Genetics* **23** (1959), 204-220.

The author studies various models which introduce an element of non-randomness into the mating between diploids in a population of fixed size, and the effect of such non-randomness on the asymptotic rate at which the population tends to homozygosity. Such models are easily constructed only for a positive degree of association between mating individuals. The probabilities of the population becoming homozygous for each of two alleles is also discussed for these systems.

P. A. P. Moran (Canberra)

6637:

Watterson, G. A. A new genetic population model, and its approach to homozygosity. *Ann. Human Genetics* **23** (1959), 221-232.

A population genetic model is constructed with fixed numbers of male and female parents, and the condition is imposed that the birth of a new individual entails the death of its parent of the same sex. The asymptotic rate of approach to homozygosity, with and without assortative mating, is obtained, the results being rather different from those for more usual models.

P. A. P. Moran (Canberra)

6638:

Bodmer, W. F. Discrete stochastic processes in population genetics. *J. Roy. Statist. Soc. Ser. B* **22** (1960), 218-244.

The evolution of a population under selection and restricted breeding systems is studied by writing down the equations for the appropriate discrete stochastic process, and taking a linear approximation. Particular attention is given to the systems (i) random mating and selection, (ii) incompatibility systems of mating and selection. This gives estimates of the chances of survival or extinction of genes in small populations. The special incompatibility system of primroses is studied deterministically and stochastically by the use of a computer, without linearization.

C. A. B. Smith (London)

6639:

Tallis, G. M. The sampling errors of estimated genetic regression coefficients and the errors of predicted genetic gains. *Austral. J. Statist.* **2** (1960), 66-77.

Two methods of estimating genetic regression coefficients are considered, one based on direct multiple regression (e.g., of offspring on parents) and one by variance and covariance component analysis (e.g., of sib or half-sib data). Formulas are derived for the error variances and covariances of the regression coefficients so calculated, and from these an error variance and approximate confidence interval is found for the economic gain from selection.

C. A. B. Smith (London)

6640:

Trucco, Ernesto. Note on a linear system of differential equations. *Bull. Math. Biophys.* **22** (1960), 169-180.

The author writes the system  $y_1' = -y_1 + y_2$ ,  $y_n' = y_{n-1} - 2y_n + y_{n+1}$  ( $n = 2, 3, \dots, m-1$ ),  $y_m' = y_{m-1} - y_m$  in matrix-vector form and gives the solution in the standard way in terms of the eigenvalues of the coefficient matrix. The system occurs in a paper by A. E. Roy [same *Bull.* **22** (1960), 139-168; MR **22** #4571] on the storing of information.

W. J. Coles (Salt Lake City, Utah)

# INFORMATION AND COMMUNICATION THEORY

See also A5882, 6005, 6035, 6092, 6093, 6095, 6099, 6578, 6584.

6641:

Jaglom, A. M. [Yaglom, A. M.]; Jaglom, I. M. [Yaglom, I. M.]. *Wahrscheinlichkeit und Information*. VEB Deutscher Verlag der Wissenschaften, Berlin, 1960. 189 pp. DM 9.60.

Translated by Dieter König from *Veroyatnost' i Informatsiya* [GITTL, Moscow, 1957; MR **19**, 990].

6642:

Drenick, R. F. Random processes in control and communications. *Science* **132** (1960), 865-870.

An expository and popular lecture on modern communication and information theory.

6643:

Vallet, Henri. *Essai d'une théorie générale de l'information*. *Cybernetica* **3** (1960), 175-215.

The author's aim is to present ideas leading up to a 'general theory of information' parallel to the thermodynamical theory of heat, and of comparable generality. The exposition is essentially non-mathematical; its degree of success must be left to the judgement of the comprehending reader.

A. Feinstein (Urbana, Ill.)

6644:

Baghdady, Elie J. (Editor). *Lectures on communication system theory*. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1961. xii + 617 pp. \$12.50.

This book contains a compendium of technical reports and papers on modern communication theory as written

by 18 eminent researchers, mostly connected with the Massachusetts Institute of Technology. The tutorial work is an outgrowth of a special summer program on "reliable long-range radio communication" offered to practicing communications engineers and held from 17-28 August, 1959 under the auspices of the Massachusetts Institute of Technology.

It has 23 chapters the contents of which roughly fall into 5 major categories: (i) introductory and extroductory remarks (2 papers); (ii) the mathematical representation of signals and disturbances (3 papers); (iii) the characterization of channels as communication transducers (5 papers); (iv) investigations of corrective measures to obtain certain prescribed channel characteristics in the presence of noise (10 papers); and (v) applications of the methods to technology (3 papers). [Reviews of individual chapters will be published separately.]

The book is another fine exemplar of the uses of statistical methods, among other things, in the physical sciences, that parallels the investigations of a recent symposium on electromagnetic propagation [W. C. Hoffman, editor, *Statistical methods in radio wave propagation*, Pergamon Press, New York, 1960; MR **22** #3420].

A. A. Mullin (Urbana, Ill.)

6645:

Turin, George L. An introduction to matched filters. *Trans. IRE IT-6* (1960), 311-329.

Author's summary: "In a tutorial exposition, the following topics are discussed: definition of a matched filter; where matched filters arise; properties of matched filters; matched-filter synthesis and signal specification; some forms of matched filters."

6646:

Blasbalg, H. Experimental results in sequential detection. *Trans. IRE IT-5* (1959), 41-51.

Author's summary: "The main body of this paper reports on experimental results in sequential detection. In particular, it is shown that the Wald theory of sequential analysis agrees well with experiment for the important case of Bernoulli detection even when the excess over the boundaries at the termination of an experiment is neglected. The design of the experiments, as well as the experimental apparatus, are also discussed. Experimental curves of the Operating Characteristic (OC) and Average Sample Number (ASN) functions for several sets of parameters are given.

"A publication relative to the main body of this paper [H. Blasbalg, *Ann. Math. Statist.* **28** (1957), 1024-1028; MR **20** #402] is summarized. The results of this publication are used in the Addendum, to study the resonant properties of the exponential class of sequential detectors. The practical use of these detectors for parameter estimation is discussed."

6647:

Jacobs, O. L. R. The measurement of mean square value of certain random signals. *J. Electronics Control* (1) **9** (1960), 149-159.

Author's summary: "Some practical methods of measuring the mean square value of a common class of random signals are analysed. It is shown that for averaging

operations an integrator is more efficient than a low pass filter. The variance of estimates is given for certain simple cases when an integrator is used in conjunction with either a squaring device or a rectifier."

6648:

Ostianu, V. M. A class of checking schemes. *Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.)* **2** (50) (1958), 319-328. (Russian)

A brief discussion of the binary coding-decoding methods of Hamming, Gavrilov, and Varshamov, followed by the construction of the decoding circuits appropriate to a very simple example. A. Feinstein (Urbana, Ill.)

6649:

Ferguson, David E. Fibonacci searching. *Comm. ACM* **3** (1960), 648.

6650:

Aki, Keiiti; Nordquist, John M. Automatic computation of impulse response seismograms of Rayleigh waves for mixed paths. *Bull. Seismol. Soc. America* **51** (1961), 29-34.

Authors' summary: "A program has been devised to compute theoretical seismograms of Rayleigh waves for a given epicenter and a given station entirely automatically on an electronic computer. The earth's surface is divided into three regions; continents, Pacific Ocean, and oceans other than the Pacific. Allowance can be made for differences in structure in these regions. This simple division seems satisfactory at present for Rayleigh waves of periods longer than 35 sec."

6651:

Adams, W. M.; Allen, D. C. Reading seismograms with digital computers. *Bull. Seismol. Soc. America* **51** (1961), 61-67.

Authors' summary: "A device that permits direct input of seismic traces into electronic digital computers is described. Examples of its use and its several merits are presented. The device makes feasible numerical analysis of data recorded in analog form on photographic film or paper."

6652:

Manfrino, Renato. L'entropia della lingua italiana ed il suo calcolo. *Alta Frequenza* **29** (1960), 4-29.

Author's summary: "Viene svolto in modo sistematico lo studio dell'entropia della lingua italiana. A tale scopo si sono effettuati esaurienti rilievi statistici sulla frequenza delle lettere, dei digrammi, dei trigrammi e delle parole in testi di vario tipo (storico, scientifico, giornalistico) compilati in italiano, per ricavarne l'andamento asintotico dell'entropia o contenuto medio d'informazione dei simboli (lettere). I risultati conseguiti vengono raffrontati con quelli ottenuti da Shannon, K  pfm  ller, Barnard ed altri per alcune delle pi   importanti lingue occidentali."

1122

## SERVOMECHANISMS AND CONTROL

See also A5573, A5597, A5598, A5599, A5600.

6653:

Booth, Andrew D. (Editor). ★Progress in automation. Vol. 1. Academic Press Inc., New York; Butterworths Scientific Publications, London; 1960. viii+231 pp. \$8.50.

This is a collection of eleven papers from British workers in the field of mechanical automation. Six papers are directed toward methods of measurement and analog to digital conversion. Four papers treat applications in machine tool control, manufacture of steel strip and automatic inspection. The introductory paper is by the editor. The work is an interesting documentation of the progress in automation in the United Kingdom. Because of the rate of progress in these areas, however, most of this work is of historical interest only.

F. W. Westervelt (Ann Arbor, Mich.)

6654:

Gnoenskiĭ, L. S. On accumulation of disturbances in nonstationary linear impulsive systems. *Prikl. Mat. Meh.* **23** (1959), 1136-1141 (Russian); translated as *J. Appl. Math. Mech.* **23**, 1627-1636.

A method is given for determining the maximum value  $y_{\max}(T)$  of the particular solution of the linear difference equation

$$y(t+n) + P_1(t)y(t+n-1) + \dots + P_n(t)y(t) = f(t)$$

at the fixed time  $T$  when constraints are imposed upon the modulus of  $f(t)$  and/or its derivatives over the interval  $0 \leq t \leq T$ . Two cases are treated: (A)  $|f^{(m)}(t)| \leq M_m$ ,  $m > 0$ , and (B)  $|f(t)| \leq M_0$ ,  $|f'(t)| \leq M_1$ ,  $|f''(t)| \leq M_2$ . This extends work of Roitenberg [*Prikl. Mat. Meh.* **22** (1958), 534-536; MR **21** #4864], who considered case (B) but without constraints on  $|f'(t)|$  and  $|f''(t)|$ .

J. F. Heyda (Cincinnati, Ohio)

6655:

Romanov, M. I. Algebraic criteria for aperiodicity of linear systems. *Dokl. Akad. Nauk SSSR* **128** (1959), 291-294 (Russian); translated as *Soviet Physics. Dokl.* **4** (1960), 955-961.

By aperiodicity the author means a mode of response associated with a system whose characteristic equation has only real negative roots. Let  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  be a polynomial with positive coefficients and no multiple zeros. Let  $\varphi(x) = f(x) - if'(x)$  and  $\varphi^*(x) = f'(x) - if(x)$ . The author shows that in order that all roots of  $f(x) = 0$  be real and negative, it is necessary and sufficient that the Hermitian form

$$K(\varphi, \varphi^*) = \frac{\varphi(x)\varphi^*(y) - \varphi(y)\varphi^*(x)}{x-y} \\ = \sum_{i,k=0}^{n-1} a_{ik} x^i y^k$$

be positive definite.

L. A. Zadeh (Berkeley, Calif.)

6656:

Mihailov, N. N. The equation and certain properties of an automatic control system's root locus. *Avtomat. i Telemekh.* **20** (1959), 1095-1102 (Russian. English summary); translated as *Automat. Remote Control* **20** (1960), 1063-1070.

Consider the equation  $w = F(p) = \sum_0^m b_i p^i / \sum_0^n a_i p^i$ . As the real parameter  $w$  changes,  $-\infty < w < \infty$ , the roots of this equation move along the "root locus" of  $w = F(p)$ . The author studies the properties of this root locus. Let  $T(m, n)$  denote the root locus for an  $F(p)$  which has  $m$  zeros and  $n$  poles. A general formula for  $T(m, n)$  is obtained from which it is derived, for example, that: (i) the root locus contains the real axis; (ii) that the remaining arms of  $T(m, n)$  are symmetric with respect to the real axis; (iii)  $2|n-m|$  of these arms (including the real axis) go to infinity and these arms have asymptotes which intersect in a common point. Other properties of  $T(m, n)$  are studied and their connection with linear automatic control systems is suggested.

J. Hartmanis (Schenectady, N.Y.)

6657:

Sawaragi, Yoshikazu; Tokumaru, Hidekatsu. Some numerical methods for analysis of transient responses of nonlinear control system. Mem. Fac. Engrg. Kyoto Univ. 22 (1960), 136-149.

The authors discuss two methods of obtaining the transient response of a nonlinear feedback control system. The first method is to use the discrete time superposition theorem, with either an analytic or graphical knowledge of the nonlinear input-output relation. The second method is to express the input and output of each system block by a power series in  $Z^{-n}$  (e.g.,  $E(Z) = \sum_{n=0}^{\infty} E_n Z^{-n}$ ). By substituting these power series in the  $Z$  transform of the system transfer function and equating coefficients of like powers of  $Z^{-n}$  a system of difference equations is established which may be solved numerically. In both methods the authors discuss the use of data extrapolators; the zero order hold, the first order (velocity) hold and the second order (acceleration) hold.

The authors analyze three simple linear systems and calculate the error obtained for each of the data extrapolators. The authors conclude with the rather obvious fact that the higher order data extrapolators are unwarranted, as increased accuracy when using the zero order hold can be obtained by increasing the sampling rate. The  $Z$ -transform method proposed seems unduly cumbersome, and even in the simple examples chosen, the unwieldiness of the method is evident. The method using the discrete time superposition integral is fundamental in concept.

R. J. Roy (Troy, N.Y.)

6658:

Yakubovič, V. A. On non-linear differential equations of automatic control systems with one control unit. Vestnik Leningrad. Univ. 15 (1960), no. 7, 120-153. (Russian. English summary)

Essentially, the author re-establishes some of the basic results of A. I. Lur'e [*Nekotorye nelineinye zadachi teorii avtomaticheskogo regulirovaniya*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1951; MR 15, 707] more rigorously, more elegantly, and more completely. The main results concern the stability in the large of the trivial solution of the system of  $n+1$  equations:

$$(1) \quad \begin{aligned} \frac{dx}{dt} &= Ax + a\varphi(\sigma), \\ \frac{d\sigma}{dt} &= b \cdot x - p\varphi(\sigma), \end{aligned}$$

where  $x$  and  $a$  are column vectors,  $b$  is a row vector,  $p$  is a constant,  $A$  is an  $n \times n$  matrix,  $\sigma$  is a real variable, and  $\varphi(\sigma)$  is a continuous function on  $(-\infty, \infty)$  satisfying the conditions  $\varphi(0) = 0$ ,  $\sigma\varphi(\sigma) > 0$  for  $\sigma \neq 0$ . In particular, the author shows that, in order that the trivial solution of (1) be stable in the large for any  $\varphi(\sigma)$  satisfying these conditions, it is sufficient (as well as necessary if the stability follows from the existence of a Lyapounov function of the form  $V(x, \sigma) = (Hx, x) + \int_0^\sigma \varphi(\sigma) d\sigma$ , where  $H$  is an  $n \times n$  Hermitian matrix) that the linear system resulting from replacing  $\varphi(\sigma)$  in (1) by  $\mu\sigma$ , where  $\mu$  is an arbitrary positive constant, have a Lyapounov function of the form  $V(x, \sigma) = (Hx, x) + \sigma^2$ , with  $H$  positive definite.

L. A. Zadeh (Berkeley, Calif.)

6659:

Merkulova, E. P. The problem of optimizing systems which contain essentially nonlinear elements. Avtomat. i Telemekh. 20 (1959), 1335-1344 (Russian. English summary); translated as Automat. Remote Control 20 (1960), 1303-1313.

The author considers a nonlinear single loop feedback system comprising fixed memoryless nonlinear two-poles, a fixed linear two-pole and an adjustable linear correcting network. The system is assumed to be subjected to a random input consisting of a stationary noise component with zero expectation and known correlation function, and a stationary signal component with non-zero expectation and known correlation function. The problem is to find a correcting network which minimizes the variance of the difference between the input and output. By using the technique of Kazakov [Avtomat. i Telemekh. 17 (1956), 385-409; MR 18, 520] and Boonton [Proc. Sympos. Nonlinear Circuit Analysis (New York, 1953), pp. 369-391, Polytechnic Institute of Brooklyn, New York, 1953; MR 16, 1036], the nonlinear elements are replaced by constants whose values depend on the mean and mean square values of their respective inputs. The resulting system is treated as if it were linear. To find the impulsive response of the correcting network the author introduces several simplifying assumptions which are not adequately justified. Eventually, the determination of the impulsive response is reduced to the solution of a system of Wiener-Hopf integral equations with a finite upper limit.

L. A. Zadeh (Berkeley, Calif.)

6660:

Fitch, Frederic B. Representation of sequential circuits in combinatory logic. Philos. Sci. 25 (1958), 263-279.

This paper gives a way of representing synchronous sequential switching circuits in terms of a generalization of the author's system of "basic logic". Although this system is used to represent sequential circuits and their behavior, it is not especially convenient to work with the circuits in this form. Certain "feedback equivalences" (so called since they describe circuit feedback loops) hold by describing differently cut feedback paths of the same circuits. One formally given feedback equivalence (and several informally described generalizations of it) state the equivalence of circuits having different numbers of elements. The author suggests the use of these equivalences to simplify circuits, but the main result is an application of circuit ideas to logic. The author adds these feedback equivalences to the axioms of his system of combinatory logic, and claims that even with the new axioms the system can be proved consistent. E. F. Moore (Murray Hill, N.J.)



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